MAXIMAL ABELIAN SUBALGEBRAS IN HYPERFINITE FACTORS

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1. **Introduction.** In this note we outline a construction which, together with a new invariant, gives still more maximal abelian subalgebras of the hyperfinite factor of type II_1. We also state some results concerning maximal abelian subalgebras of hyperfinite factors of type III. Complete proofs will appear elsewhere.

First we will establish some notation and terminology. Let $\mathfrak{M}$ and $\mathfrak{N}$ be maximal abelian subalgebras of a factor $\mathfrak{A}$. We call $\mathfrak{M}$ and $\mathfrak{N}$ equivalent (in $\mathfrak{A}$) if there is an automorphism of $\mathfrak{A}$ carrying $\mathfrak{M}$ onto $\mathfrak{N}$. Let $N(\mathfrak{M}) = N^1(\mathfrak{M})$ be the subalgebra of $\mathfrak{A}$ generated by all those unitary operators U in $\mathfrak{A}$ with $U\mathfrak{M}U^* = \mathfrak{M}$, and let $N^j(\mathfrak{M}) = N(N^{j-1}(\mathfrak{M}))$ for $j > 1$. Following Dixmier [2] and Anastasio [1], we call $\mathfrak{M}$ regular if $N(\mathfrak{M}) = \mathfrak{A}$, semiregular if $N(\mathfrak{M})$ is a factor distinct from $\mathfrak{A}$, and n-semiregular ($n \geqq 2$) if no $N^j(\mathfrak{M})$, $1 \leqq j < n$ is a factor but $N^n(\mathfrak{M})$ is. We say that $\mathfrak{M}$ has proper [improper] length n if n is the smallest positive integer such that $N^n(\mathfrak{M}) = N^{n+1}(\mathfrak{M})$ and if $N^n(\mathfrak{M}) = \mathfrak{A}$ [$N^n(\mathfrak{M}) \neq \mathfrak{A}$]. Notice that the length of a maximal abelian subalgebra is invariant under equivalence. We are now able to state our main results.

THEOREM 1. *For each choice of $n = 2, 3$ and $k = 0, 1, 2, \cdots$, the hyperfinite* II_1 *factor contains an n-semiregular maximal abelian subalgebra of improper length $n+k$.*

THEOREM 2. *Let $\mathfrak{A}$ be one of the hyperfinite type* III *factors of Powers (cf. [5]). Then $\mathfrak{A}$ contains a regular and two inequivalent semiregular maximal abelian subalgebras. Also, for each choice of $n = 2, 3$ and $k = 0, 1, 2, \cdots$, $\mathfrak{A}$ contains two n-semiregular maximal abelian subalgebras, one of proper length $n+k$ and one of improper length $n+k$.*

REMARKS. (1) These results are a summary of the author's doctoral thesis written at the University of British Columbia under the supervision of Dr. D. Bures.

(2) Anastasio has shown that Theorem 1 holds with "improper" replaced by "proper" [1]. Because of a previous remark, our subalgebras are mutually inequivalent as well as inequivalent to those of Anastasio.

(3) $\mathfrak{M}$ has proper length n if and only if $\mathfrak{M}$ has length $n-1$ in the sense of Tauer [7].

(4) Pukánszky has given a general method for constructing maximal abelian subalgebras in a wide class of type III factors [6]; but, because of an error in the proof of Lemma 17, the types of these subalgebras is not known.

2. **Construction of II_1 factors.** Let G be a countably infinite group with identity e and let Δ be the set of all functions with finite support from G into $\{0, 1\}$. Under point-wise addition modulo 2, Δ becomes an abelian group. For $g \in G$ and $\alpha \in \Delta$, define elements $g\alpha$ and αg in Δ by

$$\begin{aligned} (g\alpha)(h) &= \alpha(g^{-1}h) && (h \in G), \\ (\alpha g)(h) &= (\alpha(g))^2 - 2\alpha(g) + 1 && h = g, \\ &= \alpha(h) && \text{otherwise}. \end{aligned}$$

Let H be a Hilbert space with orthonormal basis $(\phi_\alpha)_{\alpha \in \Delta}$. For each $g \in G$, define operators F_g and U_g on H by

$$\begin{aligned} F_g\phi_\alpha &= \tfrac{1}{2}\phi_\alpha + \tfrac{1}{2}\phi_{\alpha g} && (\alpha \in \Delta), \\ U_g\phi_\alpha &= \phi_{g\alpha} && (\alpha \in \Delta). \end{aligned}$$

Notice that $\{F_g : g \in G\}$ is a commuting family of projections, that $g \to U_g$ is a unitary representation of G on H, and that $U_g F_h U_g^* = F_{gh}$ for all $g, h \in G$. It is easy to verify that $\mathfrak{M}$, the von Neumann algebra on H generated by $\{F_g : g \in G\}$, is maximal abelian in $\mathcal{L}(H)$ (use [3, p. 109, Exercise 5] and ϕ_0). Let $\mathcal{B}(G)$ be the von Neumann algebra on $H \otimes K$ generated by

$$\{(M \otimes I)(U_g \otimes V_g) : M \in \mathfrak{M}, g \in G\},$$

where K is the Hilbert space of all complex-valued square-summable functions on G and V_g is the unitary operator on K satisfying $(V_g x)(h) = x(g^{-1}h)$ for all $h \in G$, $x \in K$. Using some standard results from [3, pp. 127–137], one proves

LEMMA 3. *$\mathcal{B}(G)$ is a factor of type II_1 which is hyperfinite whenever G is the increasing union of a sequence of finite subgroups.*

3. **Subalgebras of II_1 factors.** For a subgroup G_0 of G, let $N(G_0)$ be the normalizer of G_0 in G and let

$$\mathfrak{M}(G_0) = \mathcal{R}(U_g \otimes V_g : g \in G_0).$$

Notice that $\mathfrak{M}(G_0)$ is always a proper subalgebra of $\mathcal{B}(G)$. In the following three lemmas, G_0 denotes a subgroup of G.

LEMMA 4 (CF. [6, LEMMA 14]). *$\mathfrak{M}(G_0)$ is maximal abelian in $\mathcal{B}(G)$ if $\{ghg^{-1}: g\in G_0\}$ is infinite whenever $h\in G-G_0$.*

PROOF. A straightforward calculation.

LEMMA 5. *$\mathfrak{M}(G_0)$ is a factor if and only if all nontrivial conjugate classes of G_0 are infinite.*

PROOF. Observe that $\mathfrak{M}(G_0)$ is *-isomorphic to the group operator algebra over G_0, and apply [4, Lemma 5.3.4].

LEMMA 6 (CF. [6, LEMMA 17]). *Suppose that G_0 satisfies: given a finite subset $F\subset G$ and a $g\in G$, there are infinitely many elements $g_0\in G_0$ such that*

(i) *$h, k\in F$ and $hg_0k^{-1}=g_0$ imply $h=k$,*

(ii) *if $g\notin N(G_0)$, then also $gg_0g^{-1}\notin G_0$. Then $N(\mathfrak{M}(G_0))=\mathfrak{M}(N(G_0))$.*

4. Proofs of the theorems. Applying the four preceding lemmas to the groups and subgroups used to prove Theorems I and II of [1], our Theorem 1 follows. The proof of Theorem 2, which is very involved technically, is based on the proof of [6, Lemma 17].

REFERENCES

1. S. Anastasio, *Maximal abelian subalgebras in hyperfinite factors*, Amer. J. Math. **87** (1965), 955–971.

2. J. Dixmier, *Sous-anneaux abéliens maximaux dans les facteurs de type fini*, Ann. of Math. (2) **59** (1954), 279–286.

3. ———, *Les algèbres d'opérateurs dans l'espace Hilbertion*, Gauthier-Villars, Paris, 1957.

4. F. J. Murray and J. von Neumann, *On rings of operators*. IV, Ann. of Math. (2) **44** (1943), 716–808.

5. R. T. Powers, *Representations of uniformly hyperfinite algebras and their associated von Neumann rings*, Bull. Amer. Math. Soc. **73** (1967), 572–575.

6. L. Pukánszky, *Some examples of factors*, Publ. Math. Debrecen **4** (1956), 135–156.

7. Sister R. J. Tauer, *Maximal abelian subalgebras in finite factors of type* II, Trans. Amer. Math. Soc. **114** (1965), 281–308.

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