RINGS WITH TRANSFINITE LEFT DIVISION ALGORITHM

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The aim of this note is to describe the structure of a class of noncommutative rings which possess a variant of the Euclidean algorithm and indicate some properties of such rings.

All rings are associative and possess unity; subrings and homomorphisms are unitary. A *domain* is a (not necessarily commutative) ring without nonzero zero-divisors.

Let R be a ring and ϕ be an ordinal-valued function defined on $R\sim(0)$. Put $\phi(0)=-\infty$ and let $(-\infty)+(-\infty)=\alpha+(-\infty)=(-\infty)+\alpha=-\infty$ and $-\infty<\alpha$ for every ordinal α in the range of ϕ. ϕ is called a *transfinite left division algorithm* on R if, for all $a, b\in R$, the following conditions hold:

(1) $\phi(a-b)\leqq\max\{\phi(a), \phi(b)\}$,

(2) $\phi(ab)=\phi(b)+\phi(a)$,

(3) if $b\neq 0$, then there exist $q, r\in R$ such that $a=qb+r$, $\phi(r)<\phi(b)$.

Clearly, every ring with a transfinite left division algorithm is a left principal ideal domain.

We need some terminology and notations. Let ρ be a mono-endomorphism of a domain D. A mapping $\delta: D\rightarrow D$ is called a ρ-derivation on D if $\delta(a+b)=\delta(a)+\delta(b)$ and $\delta(ab)=\rho(a)\delta(b)+\delta(a)b$ hold for all $a, b\in D$.

Let D be a subdomain of a domain R. Let x be an element of R such that every nonzero element $r\in R$ can be uniquely expressed as $\sum_{i=0}^{s} d_i x^{n_i}$ where $d_i\in D\sim(0)$ and n_i are integers with $0\leqq n_0<\cdots<n_s$. Further, suppose that there exists a mono-endomorphism ρ of D and a ρ-derivation δ on D such that $xd=\rho(d)x+\delta(d)$ for all $d\in D$. This situation is expressed symbolically as $R=D[x, \rho, \delta]$.

Let R be a domain, α a nonzero ordinal and $\{R_\beta: \beta<\alpha\}$ a set of subdomains of R such that

(1) $R=\bigcup_{\beta<\alpha} R_\beta$,

(2) if $0<\beta<\alpha$ then $R_\beta=(\bigcup_{\gamma<\beta} R_\gamma)\ [x_\beta, \rho_\beta, \delta_\beta]$. We express this situation symbolically as $R=R_0[x_\beta, \rho_\beta, \delta_\beta: 0<\beta<\alpha]$. Thus, $\bigcup_{\gamma<\beta} R_\gamma=R_0[x_\gamma, \rho_\gamma, \delta_\gamma: 0<\gamma<\beta]$. If all δ_β are zero derivations, we simplify the notation and put $R=R_0[x_\beta, \rho_\beta: 0<\beta<\alpha]$.

THEOREM 1 (CF. [2], [4]). *A ring R has a transfinite left division algorithm if and only if $R=K[x_\beta, \rho_\beta, \delta_\beta: 0<\beta<\alpha]$, where K is a skew* *field and, for every* $0<\beta<\alpha$,

$$\rho_\beta(K[x_\gamma, \rho_\gamma, \delta_\gamma: 0 < \gamma < \beta]) \subseteq K.$$

A construction is given to prove the following

THEOREM 2. *Let k be an arbitrary skew field and α be an arbitrary nonzero ordinal. There exists a skew field K containing k as a subskew field and a ring $R=K[x_\beta, \rho_\beta: 0<\beta<\alpha]$ such that*

$$\rho_\beta(K[x_\gamma, \rho_\gamma: 0 < \gamma < \beta]) \subseteq K.$$

For $\alpha=1$, any skew field would do. For $\alpha=2$, $K[x, id]$ works. For $\alpha=3$, Theorem 2 already contains a counterexample to a conjecture of I. N. Herstein, stated as highly likely to be true [3, p. 75]. For other implications, see [5].

In the following two theorems, K is a skew field and

$$R = K[x_\beta, \rho_\beta: 0 < \beta < \alpha]$$

where, for $0<\beta<\alpha$,

$$\rho_\beta(K[x_\gamma, \rho_\gamma: 0 < \gamma < \beta]) \subseteq K.$$

THEOREM 3 (CF. [1]). *R is a right primitive ring. R is a left primitive ring if and only if α is a nonlimit ordinal.*

THEOREM 4. *Let Ω_λ be the first ordinal of cardinality $\aleph_\lambda$. We have*

$$\begin{aligned} \text{r. gl. dim}\, R &= \infty && \text{if } \alpha \geqq \Omega_\omega, \\ &\geqq n+2 && \text{if } \alpha \geqq \Omega_n + 1 \quad \text{where} \quad n < \omega, \\ &\geqq 2 && \text{if } \omega \geqq \alpha > 2. \end{aligned}$$

Using Theorems 2 and 4 it is shown that there exist rings with a transfinite left division algorithm having a prescribed right global dimension. Notice that the left homological dimension of any such ring is either 0 or 1 (cf. [6 and references given there]).

In a slightly different direction, we have

THEOREM 5. *A domain $R=D[x_\beta, \rho_\beta: 0<\beta<\alpha]$ is a left principal ideal domain if and only if D is a left principal ideal domain and, for every* $0<\beta<\alpha$,

$$\rho_\beta(D[x_\gamma, \rho_\gamma: 0 < \gamma < \beta] \sim (0)) \subseteq U(D)$$

where $U(D)$ is the group of units of D.

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