A METHOD FOR COMPARING UNIVALENT FUNCTIONS

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1. The Loewner representation. Let f be any function in the class S of univalent functions on the unit disk bearing the normalization $f(0)=0, f'(0)=1$. Then it is known [1], [2], that f can be represented as

$$f(z) = \lim_{t\to\infty} e^t h(z, t), \tag{1}$$

locally uniformly in z on $|z|<1$, where $h(z, \cdot)$ is the solution of Loewner's equation in its general form

$$\frac{dh}{dt} = -hp(h, t) \qquad \text{a.e. in } t \text{ on } [0, \infty) \tag{2}$$

with the initial values

$$h(z, 0) = z, \qquad |z| < 1. \tag{3}$$

Here $p(\cdot, t)$ denotes a suitably chosen one-parameter family of holomorphic functions on the unit disk having positive real part and normalized so that $p(0, t)\equiv 1$, whose dependence on t is Lebesgue measurable on $[0, \infty)$ whenever the first variable is held fixed, and the solution of (2) is understood in the Carathéodory sense [3].

Conversely, if $p(\cdot, t)$ denotes any family of functions satisfying the above requirements, then the solution to the foregoing initial-value problem is known to exist and be holomorphic and univalent in z on the unit disk; the limit in (1) is then also known to exist and to determine a function in class S [1], [2].

More generally, one can consider the general solution $h(z, s, t)$ of (2) for $0\leqq s\leqq t$ with the initial values

$$h(z, s, s) = z, \qquad |z| < 1.$$

Then, in place of (1), one has

$$\lim_{t\to\infty} e^t h(z, s, t) = g(z, s)$$

locally uniformly in z for $|z|<1$, where $e^{-s}g(z, s)$ now belongs to S for *all* s in $[0, \infty)$. The function g is absolutely continuous in s and constitutes an integral of Equation (2), for it is easily shown that

$$g(h(z,t),t) = f(z) \tag{4}$$

for all t in $[0, \infty)$, [1], [2], and, in particular, that

$$g(z,0) = f(z).$$

2. **Comparisons, local and global, in the class S.** The identity (4) can be used as the basis for a local variational theory within the class S, by subjecting the unit disk in the h-plane to the infinitesimal transformations of the semigroup of bounded univalent functions, which Loewner has characterized [5]. This results in a variation in the function f for each fixed t, and, in e.g., the case of extremal problems for coefficients of functions in class S, it leads to a condition on the initial coefficients of the functions $p(\cdot, t)$ that generate extremal mappings that amounts to the Pontryagin maximum principle [4], and is equivalent to Schiffer's characterization of the extremal functions as solutions of his quadratic differential equation [1], [6].

What we wish to report on now is the possibility of making *global* comparisons between the functions in class S, in contrast to the local comparisons mentioned above, by considering in place of (4) the function $G(z, t)$ defined by the equation

$$g(\hat{h}(z,t),t) = G(z,t) \qquad (|z| < 1, \qquad 0 \leqq t < \infty). \tag{5}$$

Here $\hat{h}$ is understood to be the solution to the initial-value problem made up of (2) and (3) when $p(\cdot, t)$ is replaced by any other one-parameter family $\hat{p}(\cdot, t)$ that satisfies the same conditions as $p(\cdot, t)$.

The function G belongs to class S for all t in $[0, \infty)$ and satisfies

$$G(z,0) = f(z) \tag{6}$$

while (by arguments similar to those already used in [1], [2])

$$\lim_{t\to\infty} G(z,t) = \hat{f}(z), \tag{7}$$

locally uniformly in z on the unit disk, where $\hat{f}$ is the function in S generated by $\hat{h}$ in the same manner as (1). Moreover, G is absolutely continuous in t, locally uniformly in z, and its t-derivative is given by

$$\partial_t G(z,t) = \hat{h}(z,t) g'(\hat{h}(z,t),t)\left[p(\hat{h}(z,t),t) - \hat{p}(\hat{h}(z,t),t)\right] \quad \text{a.e.}, \tag{8}$$

where the $'$ denotes the derivative of g w.r.t. its first argument.

This procedure makes it possible to join a given function f in S with any other function $\hat{f}$ in S along an absolutely continuous path in S, thereby generalizing the procedure of §1, which constitutes the special case of $f(z) = z$ in the present set-up.

3. Application to extremal problems for coefficients of functions in S. If we expand both sides of (8) in power series about the origin, we find that the nth coefficient of $\partial_t G$ is given by the expression

$$\sum_{m=1}^{n-1} B_m(t)[p_m(t) - \hat{p}_m(t)] \quad \text{a.e.,} \tag{9}$$

where the p_m and $\hat{p}_m$ are the mth coefficients of $p(\cdot, t)$ and $\hat{p}(\cdot, t)$, resp., and the B_m are certain combinations of the initial coefficients of the functions g and $\hat{h}$ making them absolutely continuous as functions of t. Their derivatives involve the p_m and $\hat{p}_m$, and when $p(\cdot, t)$ coincides with $\hat{p}(\cdot, t)$ the resulting expressions reduce to a set of differential equations which are already known in the local theory [1], [6].

To establish the global extremality in S of the real part of the nth coefficient of a function f generated by (2), it would clearly be enough to show that the real part of (9) is nonpositive for a.e. t whatever the choice of the $\hat{p}_m$ (so long as they come from functions admissible in the sense of §1), for then the real part of the nth coefficient of the function G would be nonincreasing. In view of (6) and (7), the global extremal property would thereby be verified.

In practice, it is desirable to restrict the competing functions $\hat{p}(\cdot, t)$ by placing a limitation on the range of their initial coefficients. In certain cases it can be shown by using the symmetries of the coefficient body S_n for functions in S that this limitation results in no loss of generality.

4. Illustration: the case $n=3$. We shall verify the extremal property of the third coefficient of the Koebe function $f(z)=z/(1-z)^2$ in the class of functions $\hat{f}(z)=z+\hat{a}_2z^2+\hat{a}_3z^3+\cdots$ in S which have $\operatorname{Re}\hat{a}_2 \geqq 0$; an analogous result will hold for $f(z)=z/(1+z)^2$ when $\operatorname{Re}\hat{a}_2 \leqq 0$. We have $p(h, t)=(1-h)/(1+h)$ and $g(h, t)=e^t h/(1-h)^2$; if we put $\hat{p}(h, t)=1+2\sum_{m=1}^{\infty}\hat{p}_m(t)h^m$ and $\hat{h}(z, t)=e^{-t}(z+\sum_{m=2}^{\infty}\hat{b}_m(t)z^m)$ then the real part of (9) for $n=3$ becomes twice

$$\operatorname{Re}\{B_1(t)[-1-\hat{p}_1(t)] + B_2(t)[1-\hat{p}_2(t)]\}, \tag{10}$$

where

$$B_1(t) = 2\hat{b}_2(t)e^{-t} + 4e^{-2t}, \qquad B_2(t) = e^{-2t}. \tag{11}$$

We compute from (2)

$$\begin{aligned} \frac{d\hat{b}_2(t)}{dt} &= -2\hat{p}_1(t)e^{-t} \text{ a.e.,} \\ \operatorname{Re}\frac{d\hat{b}_3(t)}{dt} &= \operatorname{Re}\{-4\hat{b}_2(t)\hat{p}_1(t)e^{-t} - 2\hat{p}_2(t)e^{-2t}\} \text{ a.e.,} \end{aligned} \tag{12}$$

and therefore

$$\frac{dB_1(t)}{dt} = -B_1(t) + 4e^{-2t}[-1 - \hat{p}_1(t)] \quad \text{a.e.,} \tag{13}$$

while from (11) we deduce that $B_1(0) = 4$.

In order to make use of the assumption that Re $\hat{a}_2 \geqq 0$, we prove the following lemma.

LEMMA. *If* $\hat{a}_2 = \lim_{t\to\infty} \hat{b}_2(t)$ *has nonnegative real part, then the point* ($\hat{a}_2$, Re $\hat{a}_3$) *can be reached by a solution of* (12) *that starts at the origin when* $t=0$ *and has* Re $\hat{p}_1(t) \leqq 0$ *a.e. in* $[0, \infty)$.

PROOF. In view of (12), this is the same as saying that Re $\hat{b}_2(t)$ can be assumed to be monotone nondecreasing. Suppose it is not. Then there will be two values $\xi < \eta$ of t such that Re $\hat{b}_2(\xi) =$ Re $\hat{b}_2(\eta)$. On the interval $[\xi, \eta]$ we can, if need be, replace $\hat{p}(\cdot, t)$ by the functions (also of positive real part)

$$\begin{aligned} q(h, t) &= \tfrac{1}{2}[\hat{p}(h, t) + (\hat{p}(-\bar{h}, t))^-] \\ &= 1 + 2i \operatorname{Im} \hat{p}_1(t)h + 2 \operatorname{Re} \hat{p}_2(t)h^2 + \cdots. \end{aligned}$$

This makes Re $\hat{b}_2(t)$ constant on $[\xi, \eta]$ and leaves Im $\hat{b}_2(t)$ unchanged, while in the equation for Re $d\hat{b}_3(t)/dt$ the only change is that the term

$$-4 \operatorname{Re} \hat{b}_2(t) \operatorname{Re} \hat{p}_1(t)e^{-t}$$

is now missing. But

$$\int_{\xi}^{\eta} -4 \operatorname{Re} \hat{b}_2(t) \operatorname{Re} \hat{p}_1(t)e^{-t}dt = [\operatorname{Re} \hat{b}_2(\eta)]^2 - [\operatorname{Re} \hat{b}_2(\xi)]^2 = 0,$$

so that the missing term does not affect the value of Re $\hat{b}_3(\eta)$. By the Rising Sun Lemma, there are at most a countable number of disjoint intervals in $[0, \infty)$ where this alteration of $\hat{p}(\cdot, t)$ needs to be made, so that the altered $\hat{p}(\cdot, t)$ remains measurable in t and yields a trajectory of (12) that satisfies the assertion of the lemma.

A similar reasoning, in which $\hat{p}(h, t)$ is replaced by

$$q(h, t) = \tfrac{1}{2}[\hat{p}(h, t) + (\hat{p}(\bar{h}, t))^-],$$

shows that we can also assume that Im $\hat{p}_1(t)$ does not change sign on $[0, \infty)$. For Re $\hat{p}_1(t)$ and Im $\hat{p}_1(t)$ restricted in this way, Equation (13) and the initial condition $B_1(0) = 4$ imply that

$$\operatorname{Im} B_1(t) \operatorname{Im} \hat{p}_1(t) \leqq 0 \text{ a.e.,}$$

and

$$4e^{-2t} \leqq \operatorname{Re} B_1(t) \leqq 4e^{-t}$$

(since now $0 \leqq 1+\operatorname{Re} \hat{p}_1(t) \leqq 1$ a.e.).

To prove that (10) is nonpositive it is therefore enough to show that

$$-4[1+\operatorname{Re} \hat{p}_1(t)] \leqq \operatorname{Re} \hat{p}_2(t) - 1 \text{ a.e.} \tag{14}$$

We may restrict $\hat{p}_1(t)$ and $\hat{p}_2(t)$ to the form $\exp(i\theta(t))$, $\exp(2i\theta(t))$, resp., for $\theta(t)$ real-valued, either by appealing to Loewner's theory of slit mappings [5] or, even better, to the Carathéodory representation of $\hat{p}_1(t)$ and $\hat{p}_2(t)$ [1], [7]. Then (14) is equivalent to the inequality $-2[1+\cos \theta(t)]^2 \leqq 0$ a.e., and the monotonicity of the real part of the third coefficient of G is thereby proved.

This gives us Loewner's inequality Re $\hat{a}_3 \leqq 3$ when Re $\hat{a}_2 \geqq 0$, and at the same time shows that equality holds only when $\hat{f}$ is the Koebe function $z/(1-z)^2$. (If one inspects the real part of the second coefficient of G, it also is seen to be monotone decreasing, so the same Koebe function is extremal there, too.)

By a variant of the foregoing procedure one can prove a number of other inequalities, among which is Jenkins' inequality

$$\operatorname{Re}\left[e^{2i\phi}(a_3-a_2^2)-\lambda e^{i\phi}a_2\right] \leqq 1+3\lambda^2/8+(\lambda^2 \log 4/\lambda)/4$$

for ϕ real and $0<\lambda \leqq 4$ [8].

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