MAPPING CYLINDERS AND THE ANNULUS CONJECTURE

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Suppose f is an embedding of the n-sphere S^n into the $(n+1)$-sphere S^{n+1}; f is said to be *locally flat* at $x \in S^n$ if there is a neighborhood U of $f(x)$ in S^{n+1} such that the pair $(U, U \cap f(S^n))$ is homeomorphic to (E^{n+1}, E^n) where E^i is Euclidean i-space; i.e., there exists a homeomorphism $h: U \to E^{n+1}$ such that $h(U \cap f(S^n)) = E^n \equiv E^n \times 0 \subseteq E^n \times E^1 = E^{n+1}$. Brown [2], [3] has shown that if f is locally flat at each point of S^n, then the closure of each complementary domain of $f(S^n)$ in S^{n+1} is homeomorphic to an $(n+1)$-cell. One of the outstanding unsolved problems in topology of manifolds is the annulus conjecture. Suppose f, g are two locally flat embeddings (i.e., f and g are locally flat at each point of S^n) of S^n into S^{n+1} such that $f(S^n) \cap g(S^n) = \emptyset$. The connected submanifold A^{n+1} of S^{n+1} whose boundary is $f(S^n) \cup g(S^n)$ is called a *pseudo-annulus*. The annulus conjecture is that A^{n+1} is homeomorphic to $S^n \times [0, 1]$. If f, g are both either piecewise linear or differentiable maps or if $n \leqq 2$, then the conjecture is true.

This paper was motivated by an attempt to construct a counterexample to the annulus conjecture. Let $p: S^n \to S^n$ be a continuous map. The *mapping cylinder* of p, Map (p), is the decomposition space formed from the disjoint union $(S^n \times [0, 1]) \cup S^n$ by identifying $(x, 1)$ with $p(x)$ for each $x \in S^n$. The idea was to find a map $p: S^n \to S^n$ such that Map (p) is an $(n+1)$-manifold which is not homeomorphic to $S^n \times I$; for example, one might attempt to construct such a p by using a variation of Bing's example [1] of an upper semicontinuous decomposition of S^3 which yields S^3 but some of whose nondegenerate elements are spheres. By Proposition 2, Map (p) would be a pseudo-annulus and hence a counterexample. However, we show that this is impossible in dimension 3; i.e., if Map (p) is a manifold, then it is homeomorphic to $S^3 \times I$.

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Let $p: S^n \to S^n$ be a continuous map such that Map (p) is an $(n+1)$-manifold. Let $\pi: (S^n \times I) \cup S^n \to$ Map (p) be the natural projection.

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PROPOSITION 1. *The boundary of Map* (p) *is the union of the two* n*-spheres* ($S^n \times 0$) *and* ($S^n \times 1$).

PROOF. Suppose M has one boundary component. Note that M is homotopically equivalent to S^n. In the exact sequence

$$H_n(\partial M) \xrightarrow{i_*} H_n(M) \rightarrow H_n(M, \partial M)$$

i_* is the zero map and by Poincaré Duality, $H_n(M, \partial M)$ is isomorphic to $H^1(M) = 0$. Hence $H_n(M) = 0$, a contradiction.

PROPOSITION 2. *Map* (p) *is a pseudo-annulus.*

PROOF. By attaching an $(n+1)$-cell to each boundary component of M, one obtains a closed manifold S. It is easy to see that S is the union of two open $(n+1)$-cells and hence by [2], S is an $(n+1)$-sphere in which M appears as a pseudo-annulus.

PROPOSITION 3. *If* $n \neq 4$, *then* p *is a cellular map; i.e., if* $x \in S^n$, *then* $p^{-1}(x) = \bigcap_{i=1}^{\infty} C_i$ *where* C_i ($\subseteq$*interior* C_{i-1}) *are closed* n*-cells in* S^n.

PROOF. Let U be a contractible open subset of S^n. Define $g: p^{-1}U \rightarrow U$ by $g = p \mid p^{-1}U$. Map (g) is a contractible open subset of Map (p) for Map (g) $= r^{-1}U$ where r is the canonical deformation retraction of Map (p) onto image (p) $= S^n$. Thus Map (g) is an $(n+1)$-manifold. Since U is collared in Map (g) [3], Map (g) $- U$ is contractible. But Map (g) $- U$ deformation retracts to $p^{-1}U$ and hence $p^{-1}U$ is contractible. By Lacher [5, Theorem 2] for any open subset V of S^n, $p: p^{-1}(V) \rightarrow V$ is a proper homotopy equivalence. Let $x \in S^n$, then $x = \bigcap_{i=1}^{\infty} D_i$ where D_i ($\subseteq$ interior D_{i-1}) are closed n-cells in S^n. Since $p^{-1}(x) = \bigcap_{i=1}^{\infty} p^{-1}D_i$, if we want to show that $p^{-1}(x)$ is cellular, it is sufficient to show that there exists an n-cell C_i in p^{-1} (interior D_i) for each i such that $p^{-1}(x)$ is contained in the interior of C_i. From above $p: p^{-1}$ (interior D_i)$\rightarrow$interior D_i is a proper homotopy equivalence; since interior D_i is 1-connected at infinity, p^{-1} (interior D_i) is 1-connected at infinity. For $n = 3$, p^{-1} (interior D_i) is an open 3-cell by Edwards [4]. For $n \geqq 5$, we apply Stallings [7]. It is now easy to find C_i.

THEOREM. *If* $p: S^3 \rightarrow S^3$ *is a continuous map and Map* (p) *is a manifold, then Map* (p) *is homeomorphic to* $S^3 \times I$.

PROOF. By Proposition 3, p is a cellular map. By Price [6] there exists a pseudo-isotopy $H: S^3 \times I \rightarrow S^3 \times I$ (i.e., H is level preserving and the map $H_t: S^3 \rightarrow S^3$, defined by $H(x, t) = (H_t(x), t)$, is a homeomorphism for $t \in [0, 1)$) such that H_0 is the identity map and $H_1 = p$.

Define ϕ: Map $(p) \to S^3 \times I$ by $H\pi^{-1}(x)$. It is easily seen that ϕ is a homeomorphism using the fact that π is an open map.

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