AN INTEGRAL IN TOPOLOGICAL SPACES[1]

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A definition of an integral by majorants and minorants was first given by Perron in [8] for functions of one real variable. Perron's definition was generalized by Bauer in [1] and later on by Mařík in [7], for the case of functions of several real variables whose domains of definition are compact intervals. This generalization, however, was not sufficient, because no adequate substitution theorem holds: even a very simple transformation, e.g., a rotation, of a compact interval is not necessarily a compact interval again. A generalization which admits a relatively suitable substitution theorem was given by the author in [9] and [10]. Although the integral there is defined in a locally compact first countable Hausdorff space, the main emphases are on applications to Euclidean spaces; also only functions with compact domains of definition are integrated.

In this paper we shall define a Perron-like integral in an arbitrary topological space and without any restrictions on the domains of integrable functions. Such generality, of course, causes some changes in basic definitions. Because of omission of the first axiom of countability, the derivate has to be defined by convergence of nets rather than convergence of sequences. Also the possibility of noncompact domains of integration requires a different definition of the majorant.

Throughout P is a topological space and $P^{\sim} = P \cup (\infty)$ is a one-point compactification of P. If $A \subset P^{\sim}$, A^{-} and $A^{\sim}$ denote the closure of A in P and $P^{\sim}$, respectively. For $x \in P^{\sim}$, Γ_x is a local base at x in $P^{\sim}$ (see [6, p. 50]).

Let σ be a nonempty system of subsets of P such that for every $A, B \in \sigma$, $A \cap B \in \sigma$ and $A - B = \bigcup_{i=1}^{n} C_i$ where $C_1, \cdots, C_n$ are disjoint sets from σ.[2] We shall assume that $\Gamma_x \subset \sigma$ for every $x \in P$ and that for each $U \in \Gamma_\infty$ there are disjoint sets $U_{1,\infty}, \ldots, U_{p,\infty}$ from σ such that $U \cap P = \bigcup_{i=1}^{p} U_{i,\infty}$ where the integer $p \geq 1$ is independent of U.

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[2] Such a system σ is called a prering in [2]. The sets from σ generate the ring over which the integral will be defined. Since our main task is to define a nonabsolutely convergent integral, it should be mentioned here that the system σ must not be too large; e.g., if σ is a σ-ring, then usually only an absolutely convergent integral is obtained. We also note that more general systems than prerings can be used for σ, e.g., a system of all simplexes in a given Euclidean space.

The algebra generated by σ is denoted by $\sigma^{\wedge}$; clearly $P\in\sigma^{\wedge}$. If δ is a collection of subsets of P and $A\subset P^{\sim}$, we let $\delta_A=\{B\in\delta: B\subset A\}$.

A system $\delta\subset\sigma$ is said to be *semihereditary* if and only if $\sigma_0\cap\delta\neq\emptyset$ for every finite disjoint collection $\sigma_0\subset\sigma$ whose union belongs to δ. A system $\delta\subset\sigma$ is said to be *stable* if and only if $\emptyset\notin\delta$ and for every $A\in\delta$ and every $x\in P^{\sim}$ there is a $U\in\Gamma_x$ such that $\delta_{A-U}\neq\emptyset$.

By a function we shall mean an extended real-valued function. A function F defined on $\delta\subset\sigma^{\wedge}$ is said to be *superadditive* or *additive* whenever $F(\bigcup_{i=1}^n A_i)\geqq\sum_{i=1}^n F(A_i)$ or $F(\bigcup_{i=1}^n A_i)=\sum_{i=1}^n F(A_i)$, respectively, for every disjoint collection $A_1,\cdots,A_n$ from δ for which $\bigcup_{i=1}^n A_i\in\delta$ and $\sum_{i=1}^n F(A_i)$ has meaning.

Semihereditary and stable systems and their connection with superadditive functions were investigated in [11], [12], and [15].

From now on we shall assume that there is given a nonnegative additive function G defined on $\sigma^{\wedge}$ and such that $G(U)<+\infty$ for every $U\in\bigcup\{\Gamma_x: x\in P\}$. Intuitively, G is a finitely additive locally finite measure in P.

With every point $x\in P^{\sim}$ we associate a certain family κ_x of nets $\{B_U, U\in\Gamma, \subset\}\subset\sigma$ where Γ is a cofinal subset of Γ_x (see [6, Chapter 2]). The collection $\kappa=\{\kappa_x: x\in P^{\sim}\}$ is called a *convergence* (or sometimes a *derivation basis*—see [4, 1.1]). For $x\in P^{\sim}$ and $\delta\subset\sigma$, $\kappa_x(\delta)=\{\{B_U\}\in\kappa_x: \{B_U\}\subset\delta\}$.

Throughout we shall assume that the convergence κ satisfies the following conditions:[3]

$\mathcal{K}_1$. For every $x\in P$, $\{U, U\in\Gamma_x, \subset\}\in\kappa_x$ and for every integer i, $1\leqq i\leqq p$, $\{U_{i,\infty}, U\in\Gamma_\infty, \subset\}\in\kappa_\infty$.

$\mathcal{K}_2$. If $x\in P^{\sim}$ and $\{B_U, U\in\Gamma, \subset\}\in\kappa_x$, then for every $V\in\Gamma_x$ there is a $U_V\in\Gamma$ such that $B_U\subset V$ for all $U\in\Gamma$ for which $U\subset U_V$.

$\mathcal{K}_3$. If $x\in P^{\sim}$, $\{B_U\}_{U\in\Gamma}\in\kappa_x$, and Γ' is a cofinal subset of Γ, then also $\{B_U\}_{U\in\Gamma'}\in\kappa_x$.

$\mathcal{K}_4$. If $x\in P^{\sim}$, $\{B_U\}\in\kappa_x$, and $A\in\sigma$, then also $\{B_U\cap A\}\in\kappa_x$.

$\mathcal{K}_5$. If $\delta\subset\sigma$ is a nonempty semihereditary, stable system, then the set $\{x\in P^{\sim}: \kappa_x(\delta)\neq\emptyset\}$ is uncountable.

If for every $x\in P^{\sim}$, κ_x consists of all nets $\{B_U\}$ which satisfy condition $\mathcal{K}_2$, then the convergence κ is called the *natural convergence* and is denoted by κ^0. It follows from [12, 2.4] that the natural convergence already satisfies all conditions $\mathcal{K}_1$–$\mathcal{K}_5$. Therefore, conditions

[3] The abstract notion of a convergence is the basis for the whole theory. Proposition 4 indicates that we should use a convergence κ with κ_x as small as possible. On the other hand, it is plain that if the κ_x are too small, we shall not obtain any integral at all. Conditions $\mathcal{K}_1$–$\mathcal{K}_5$ form a set of minimal requirements which a convergence κ must satisfy in order to give a meaningful integral.

$\mathcal{K}_1$–$\mathcal{K}_5$ are not contradictory and it was shown in [17], that they are also independent. Other useful examples of convergences which satisfy conditions $\mathcal{K}_1$–$\mathcal{K}_5$ can be found in [9], [10], and [17].

Let $x \in P^{\sim}$, $A \subset P$, and let F be a function on σ_A. We call the number ${}_{\sharp}F(x, A) = \inf\{\liminf F(B_U): \{B_U\} \in \kappa_x(\sigma_A)\}$ the *lower limit* of F at x relative to A and the number ${}_{*}F(x, A) = {}_{\sharp}(F/G)(x, A)$[4] the *lower derivate* of F at x relative to A.

Let $A \in \sigma$ and let f be a function on A^-. A superadditive function M on σ_A is said to be a *majorant* of f on A if and only if there is a countable set $Z_M \subset A^-$ such that ${}_{\sharp}(-G)(x, A) \geqq 0$ for all $x \in Z_M$, ${}_{\sharp}M(x, A) \geqq 0$ for all $x \in Z_M \cup (\infty)$, and $-\infty \neq {}_{*}M(x, A) \geqq f(x)$ for all $x \in A^- - Z_M$. The number $I_u(f, A) = \inf M(A)$ where the infimum is taken over all majorants of f on A is called the *upper integral* of f over A.

PROPOSITION 1. *Let $A \in \sigma$ and let f be a function on A^-. If $f \geqq 0$ then $I_u(f, A) \geqq 0$.*

PROPOSITION 2. *Let $A \in \sigma$ and let f be a function on A^-. Then the function $I_u(f, \cdot)$ is additive on σ_A.*

PROPOSITION 3. *Let $A \in \sigma$ and let $\{f_n\}$ be an increasing sequence of functions defined on A^-. If $I_u(f_1, A) > -\infty$ then*

$$\lim I_u(f_n, A) = I_u(\lim f_n, A).$$

DEFINITION. Let $A \in \sigma^{\wedge}$ and let f be a function on A^-. Choose disjoint sets $A_1, \cdots, A_n$ from σ such that $\bigcup_{i=1}^n A_i = A$ and set

$$I_u(f, A) = \sum_{i=1}^n I_u(f, A_i).$$

When $I_u(f, A) = -I_u(-f, A) \neq \pm\infty$ this common value is called the *integral* of f over A and it is denoted by $I(f, A)$.

By [12, (1.1)] the sets $A_1, \cdots, A_n$ always exist and by Proposition 2 the value of $I_u(f, A)$ is independent of their choice. Thus the previous definition extends the original meaning of the upper integral. If $A \in \sigma^{\wedge}$, $\mathfrak{P}(A)$ denotes the family of all functions f defined on A^- for which the integral $I(f,A)$ exists and $\mathfrak{P}_0(A) = \{f \in \mathfrak{P}(A): |f| \in \mathfrak{P}(A)\}$.

THEOREM 1. *If $A \in \sigma^{\wedge}$ and $f \in \mathfrak{P}(A)$, then $f \in \mathfrak{P}(B)$ for every $B \in \sigma_A^{\wedge}$.*

THEOREM 2. *Let $A \in \sigma^{\wedge}$, f, $g \in \mathfrak{P}(A)$ and let a, b be real numbers. If*

[4] We let $a/0 = +\infty$ for $a \geqq 0$, $a/0 = -\infty$ for $a < 0$, and $a/(\pm\infty) = 0$.

$h(x)=af(x)+bg(x)$ for all $x\in A^-$ for which $af(x)+bg(x)$ has meaning, then $h\in\mathfrak{P}(A)$ and

$$I(h, A) = aI(f, A) + bI(g, A).$$

Moreover, if $f, g\in\mathfrak{P}_0(A)$ then also $h\in\mathfrak{P}_0(A)$.

COROLLARY. *Let $A\in\sigma$ and $f, g\in\mathfrak{P}_0(A)$. Then* $\max(f, g)\in\mathfrak{P}_0(A)$ *and* $\min(f, g)\in\mathfrak{P}_0(A)$. *Moreover, if $h\in\mathfrak{P}(A)$ and $f\leqq h\leqq g$, then $h\in\mathfrak{P}_0(A)$.*

THEOREM 3. *Let $A\in\sigma$, f, g, $h_n\in\mathfrak{P}(A)$, and let $f\leqq h_n\leqq g$ for $n=1, 2, \cdots$. If* $\lim h_n=h$ *then $h\in\mathfrak{P}(A)$ and*

$$I(h, A) = \lim I(h_n, A).$$

EXAMPLE 1. Let P be r-dimensional Euclidean space and let σ be the system of all half-open intervals (bounded, unbounded, or degenerate). For $x\in P^{\sim}$ let κ_x consist of all sequences $\{K_n\}_{n=1}^{\infty}\subset\sigma$ which satisfy condition $\mathcal{K}_2$ and such that either $x\in\bigcap_{n=1}^{\infty} K_n^{\sim}$ or $K_n=\emptyset$ for all sufficiently large n. Then the convergence κ satisfies conditions $\mathcal{K}_1$–$\mathcal{K}_5$ (see [12, 3.2]) and the integral I coincides with the integral defined by Mařík in [7]. In particular, if $r=1$ and if G is the restriction of the Lebesgue measure, then the integral I coincides with the classical Perron integral (see [8] or [18]).

EXAMPLE 2. Let P be the discrete space of positive integers, let σ consist of the empty set, singletons (n), and intervals $[n, +\infty)$, $n=1, 2, \cdots$, and let $\kappa=\kappa^0$ be the natural convergence. If G is the counting measure on σ, then $f\in\mathfrak{P}(P)$ if and only if the series $\sum_{n=1}^{\infty} f(n)$ is *conditionally* convergent; $I(f, P)=\sum_{n=1}^{\infty} f(n)$ when either side has meaning.

Given another convergence $\kappa'=\{\kappa_x' : x\in P^{\sim}\}$ which satisfies conditions $\mathcal{K}_1$–$\mathcal{K}_5$, we can introduce the symbols I' and $\mathfrak{P}'$ the meaning of which is obvious. The connection between I, $\mathfrak{P}$ and I', $\mathfrak{P}'$ is given by the following proposition.

PROPOSITION 4. *Let $\kappa_x'\subset\kappa_x$ for all $x\in P^{\sim}$. Then for every $A\in\sigma$, $\mathfrak{P}(A)\subset\mathfrak{P}'(A)$ and $I(f, A)=I'(f, A)$ for all $f\in\mathfrak{P}(A)$.*

A point $x\in P^{\sim}$ is said to be *simple* if and only if $\Gamma_x=\{U_n\}_{n=1}^{\infty}$, $\sigma_{U_1}=\hat{\sigma}_{U_1}$, and for every $\{B_n\}$ and $\{C_n\}$ from $\kappa_x(\sigma_{U_1})$ also $\{B_n\cup C_n\}$ and $\{B_n-C_n\}$ belong to $\kappa_x(\sigma_{U_1})$. Notice that if $\sigma=\sigma^{\wedge}$ and $\kappa=\kappa^0$ is the natural convergence, then $x\in P^{\sim}$ is simple whenever $\Gamma_x=\{U_n\}_{n=1}^{\infty}$.

THEOREM 4. *Let $A\in\sigma^{\wedge}$ and let f be a function on A^-. Let $x\in P^{\sim}$ be a simple point at which $P^{\sim}$ is Hausdorff*[5] *and let either $x\notin A^-$ or*

[5] This means that for every $y\in P^{\sim}-(x)$ there are disjoint sets $U\in\Gamma_x$ and $V\in\Gamma_y$.

$\imath(-G)(x, A) \geqq 0$. *Further suppose that* $f \in \mathfrak{P}(B)$ *for every* $B \in \hat{\sigma_A}$ *for which* $x \notin B^{-}$ *and that*

$$\lim I(f, A - B_n) = c \neq \pm \infty$$

for every $\{B_n\} \in \kappa_x$ *for which* $x \notin (A - B_n)^{-}$, $n = 1, 2, \cdots$. *Then* $f \in \mathfrak{P}(A)$ *and* $I(f, A) = c$.

For the remainder of this paper we shall assume that P is a locally compact Hausdorff space. Unless otherwise specified our terminology concerning measures is that of [3]. If $A \subset P$, χ_A denotes the characteristic function of A. Let $\mathfrak{X}$ be the family of all sets $A \subset P$ for which $\chi_{A \cap C} \in \mathfrak{P}_0(P)$ for every compact set $C \subset P$. For $A \in \mathfrak{X}$ we let $\iota(A) = I_u(\chi_A, P)$.

THEOREM 5. *The triple* $(P, \mathfrak{X}, \iota)$ *is a complete measure space and* $\mathfrak{X}$ *contains all open subsets of* P. *The measure* ι *is inner regular on open sets and outer regular and finite on compact sets. Moreover, if* $C \subset P$ *is compact, then* $\iota(C) = \inf G(A)$ *where the infimum is taken over all sets* $A \in \hat{\sigma}$ *whose interior contains* C.[6]

THEOREM 6. *A function f belongs to* $\mathfrak{P}_0(P)$ *if and only if the Lebesgue integral* $\int_P f d\iota$ *exists;* $I(f, P) = \int_P f d\iota$ *when either side has meaning. Moreover, if for every* $A \in \sigma$ *and for every function f on* A^{-} *the upper integral* $I_u(f, A)$ *is independent of values which f takes on* $A^{-} - A$, *then* $\sigma \subset \mathfrak{X}$ *and a function f belongs to* $\mathfrak{P}_0(A)$, $A \in \sigma$, *if and only if the Lebesgue integral* $\int_A f d\iota$ *exists;* $I(f, A) = \int_A f d\iota$ *when either side has meaning.*

For $A \subset P$ let $\iota_0(A) = \inf \iota(U)$, where the infimum is taken over all open sets $U \subset P$ for which $A \subset U$. It is easy to see that ι_0 is an outer measure in P and we shall denote by $\mathfrak{X}_0$ the family of all ι_0-measurable subsets of P.

THEOREM 7. *The triple* $(P, \mathfrak{X}_0, \iota_0)$ *is a complete measure space and the measure* ι_0 *is regular.*[7] *Moreover, if* $(P, \mathfrak{A}, \mu)$ *is a measure space with a regular measure* μ, $\sigma \subset \mathfrak{A}$, *and* $G(A) = \mu(A)$ *for every* $A \in \sigma$ *for which* A^{-} *is compact, then* $\mathfrak{A} \subset \mathfrak{X}_0$ *and* $\mu(A) = \iota_0(A)$ *for every* $A \in \mathfrak{A}$.

THEOREM 8. *Always* $\mathfrak{X}_0 \subset \mathfrak{X}$ *and* $\iota_0(A) = \iota(A)$ *for every set* $A \in \mathfrak{X}_0$ *which is* ι_0-σ-*finite. If P is paracompact then* $\iota_0(A) = \iota(A)$ *for every* $A \in \mathfrak{X}_0$.

THEOREM 9. *Suppose that* $\sigma \subset \mathfrak{X}_0$ *and that* $G(A) = \iota_0(A)$ *for every* $A \in \sigma$

[6] The third part of Theorem 5 is due to W. J. Wilbur.

[7] See [5, (12.39), p. 177].

for which A^- is compact. Let $A \in \sigma$ and let f be a function on A^- for which the Lebesgue integral $\int_A f d\iota_0$ exists. If $A^- - A$ is ι_0-σ-finite then $f \in \mathfrak{P}_0(A)$ and $I(f, A) = \int_A f d\iota_0$.

COROLLARY. *Let $\sigma \subset \mathfrak{T}_0$ and let $G(A) = \iota_0(A)$ for every $A \in \sigma$ for which A^- is compact. If ι_0 is σ-finite then for every $A \in \sigma$ and for every function f on A^- the upper integral $I_u(f, A)$ is independent of values which f takes on $A^- - A$.*

Let $\kappa = \kappa^0$ be the natural convergence. It was proved in [**17**] that in this case $(P, \mathfrak{T}, \iota) = (P, \mathfrak{T}_0, \iota_0)$ and every function $f \in \mathfrak{P}(A)$ is ι-measurable. Moreover, if $\sigma = \sigma^{\wedge}$ then $\mathfrak{P}(A) = \mathfrak{P}_0(A)$ for every $A \in \sigma$.

EXAMPLE 3. Let $P = (-\infty, +\infty)$ and let σ consist of all intervals which either do not contain 0 in their closures or are of the form $(-\epsilon, \epsilon)$ where $\epsilon > 0$. Let $\kappa_x = \kappa_x^0$ for $x \neq 0$ and let κ_0 consist of the net $\{(-\epsilon, \epsilon)\}_{\epsilon > 0}$ and the trivial net $\{\emptyset\}$. Finally, let G be the restriction of the Lebesgue measure. Then κ satisfies conditions $\mathcal{K}_1$–$\mathcal{K}_5$ and f belongs to $\mathfrak{P}_0(P)$ if and only if the Lebesgue integral $\int_{-\infty}^{+\infty} f$ exists; $I(f, P) = \int_{-\infty}^{+\infty} f$ when either side has meaning. On the other hand, f belongs to $\mathfrak{P}(P)$ if and only if the *Cauchy principle value*

$$(pv)\int_{-\infty}^{+\infty} f = \lim_{\epsilon \to 0+} \left(\int_{-\infty}^{-\epsilon} f + \int_{\epsilon}^{+\infty} f \right)$$

exists; $I(f, P) = (pv)\int_{-\infty}^{+\infty} f$ when either side has meaning. Notice that 0 is not a simple point of P.

Complete proofs of the results stated here will be given in [**13**], [**14**] and [**16**].

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ON GENERALIZED COMPLETE METRIC SPACES

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The following remarks are of interest in connection with the research announcement [1]:

LEMMA. *A generalized metric space is the disjoint union of metric spaces such that each metric space is infinitely distant from every other metric space.*

PROOF. Note that $d(x, y) < \infty$ is an equivalence relation, and the equivalence classes obtained are metric spaces. Also, if the generalized space is complete, so is each metric space. Q.E.D.

Let $M = \bigvee_{\alpha \in A} M_\alpha$ denote the above partitioning. The Banach contraction principle becomes

PROPOSITION 1. *Let T be a strict contraction of a generalized complete metric space $M = \bigvee_{\alpha \in A} M_\alpha$, $0 \leqq q < 1$, $d(x, y) < \infty \Rightarrow d(Tx, Ty) \leqq qd(x, y)$. For each $\alpha \in A$, $\exists \beta \in A$ such that $T(M_\alpha) \subseteq M_\beta$. There is a unique periodic point of order n in each M_α such that $T^n(M_\alpha) \subseteq M_\alpha$.*

PROOF. Let $x, y \in M_\alpha$, $Tx \in M_\beta$. Then $d(x, y) < \infty \Rightarrow d(Tx, Ty) < \infty \Rightarrow Ty \in M_\beta$. Since T^n is a strict contraction of the complete metric space M_α, it has a unique fixed point, which is a periodic point of order n for T. Q.E.D.

The local contraction principle becomes