4. I. N. Sneddon, *Special functions of mathematical physics and chemistry*, Edinburgh (1961), 162.
5. Brij Mohan, *Formulae connecting self reciprocal functions*, Indian J. Phys. 15 (1941), 337–341.
6. W. T. Howell, *On some operational representations*, Philos. Mag. 24 (1937), 1082–1083.
7. V. V. L. N. Rao, *A property of self reciprocal functions*, Vijnana Parishad Anusandhan Patrika (1) 2 (1959), 51–54.

BHOLAKPUR, SECUNDERABAD-3, A. P., INDIA.

REPRESENTATION OF NONLINEAR TRANSFORMATIONS ON L^p SPACES

BY V. J. MIZEL

Communicated by Henry McKean, July 11, 1968

This note describes integral representations obtained for a class of nonlinear functionals and nonlinear transformations on the spaces $L^p(T)$ $(1 \leqq p \leqq \infty)$ associated with an arbitrary σ-finite measure space (T, Σ, μ). The class of functionals considered here differs from those considered in [1], [3], [7], [8], [9] and its study is mainly motivated by its close connection with nonlinear integral equations [6].

In the study of nonlinear integral equations there is a fundamental class of nonlinear transformations, called Urysohn operators [6], taking measurable functions to measurable functions and having the form

$$(Ax)(s) = \int_T \phi(s; x(t), t)dt \tag{1}$$

where S, T are Lebesgue measurable subsets of R^n and $\phi: S\times R\times T \to R$ is a real valued function which is measurable on $S\times T$ for each fixed value of its second argument and continuous on R for almost all arguments in $S\times T$. An important subclass of (1) consists of those Urysohn operators whose range is in $C(S)$ where S is compact. This subclass includes the case in which the kernel ϕ is independent of its first argument, so that the operator reduces to a real valued functional:

$$F(x) = \int_T \phi(x(t), t)dt. \tag{2}$$

Functionals of the form (2) also play an important role in the theory of generalized random processes in probability [5].

Our main result gives an abstract characterization, for all σ-finite measure spaces $T=(T, \Sigma, \mu)$ and all compact Hausdorff spaces, of nonlinear transformations $A: L^p(T) \to C(S)$, $1 \leqq p \leqq \infty$, which have the form (1). In particular we characterize functionals on $L^p(T)$ having the form (2). This latter characterization extends earlier results [7], [8] involving functionals of the form

$$F(x) = \int_T \phi(x(t))d\mu \tag{3}$$

defined on (essentially) nonatomic σ-finite measure spaces. Detailed proofs and related results will be given elsewhere.

Hereafter $T=(T, \Sigma, \mu)$ will denote a σ-finite measure space and $M(T)$ will denote the class of real valued measurable functions on T.

DEFINITION. A real valued function $\phi: R \times T \to R$ is said to be of *Caratheodory* type for T, denoted $\phi \in \mathrm{Car}(T)$, if it satisfies

(i) $\phi(\cdot, t): R \to R$ is continuous for almost all $t \in T$,

(ii) $\phi(c, \cdot): T \to R$ is measurable for all $c \in R$.

Since for each simple function x, the function $\phi \circ x$ defined by $(\phi \circ x)(t) = \phi(x(t), t)$ is in $M(T)$, it follows by taking limits of sequences of simple functions and using (i) that for every $x \in M(T)$, $\phi \circ x$ is also in $M(T)$.

DEFINITION. Given a number p, $1 \leqq p \leqq \infty$, a function $\phi \in \mathrm{Car}(T)$ is said to be in the *Caratheodory p-class* for T, denoted $\phi \in \mathrm{Car}^p(T)$, if it satisfies

$$\phi \circ x \in L^1(T) \qquad \text{for } x \in L^p(T).$$

[For the case of a finite nonatomic T it is known [6, p. 27] that ϕ is in $\mathrm{Car}^p(T)$, $1 \leqq p < \infty$, if and only if

$$|\phi(x, t)| \leqq a(t) + b|x|^p \text{ for some } a \in L^1(T).]$$

THEOREM. *Let $T=(T, \Sigma, \mu)$ be a finite measure space. Let F be a real valued functional on $L^p(T)$, $1 \leqq p \leqq \infty$, which satisfies*

(i) *$F(x+y) - F(x) - F(y) = \text{const.} = c_F$ whenever $xy = 0$ a.e.,*

(ii) *F is uniformly continuous relative to L^∞ norm on each bounded subset of $L^\infty(T)$,*

(iii) *F is continuous relative to L^p norm, if $p < \infty$, and is continuous with respect to bounded a.e. convergence, if $p = \infty$.*

Then there exists a function $\phi \in \mathrm{Car}^p(T)$ such that

$$F(x) = -c_F + \int_T \phi \circ x \, d\mu \qquad \text{for } x \in L^p(T). \tag{*}$$

Moreover ϕ *can be taken to satisfy*

(a) $$\phi(0, \cdot) = 0 \ a.e.$$

and is then unique up to sets of the form $R \times N$ *with* N *a null set in* T.

Conversely, for every $\phi \in \mathrm{Car}^p(T)$ *satisfying* (a), *and for every* $c_F \in R$, (*) *defines a functional satisfying* (i), (ii) *and* (iii).

The above result extends to σ-finite measure spaces. For $p=\infty$ it remains valid as is. For $p<\infty$ it is valid if the phrase 'bounded subset of $L^\infty(T)$' is replaced by 'bounded subset of $L^\infty(T)$ which is supported by a set of finite measure.'

The proof occurs in two parts. First we consider the case $p=\infty$. The observation that $F_1 = F + c_F$ is a functional of the same type with $c_{F_1} = 0$ permits a reduction to the case $c_F = 0$. The construction of ϕ from F now depends on the fact that for each real number h the set function ν_h defined by

$$\nu_h(E) = F(h\chi_E),$$

where χ_E denotes the characteristic function of $E \in \Sigma$, is by (i) and (iii) a μ-continuous measure on T. Hence by the Radon-Nikodym theorem there exists for each h a density $\phi_h = \phi(h, \cdot)$ corresponding to ν_h. The proof that the family $\{\phi_h \mid h \in R\}$ defines a function $\phi \in \mathrm{Car}^\infty(T)$ is a generalization of the classical proof of existence of a Lebesgue set for each function $f \in L^1(R)$. In fact the classical argument corresponds to the particular ϕ given by $\phi(x, t) = |x - f(t)|$. The validity of the representation (*) is then established by use of the Vitali convergence theorem.

The proof of the converse involves a modification of Nemytskii's argument for demonstrating that for any $\phi \in \mathrm{Car}(T)$ the mapping $x \rightarrow \phi \circ x$ preserves convergence in measure [6], together with the Banach-Saks theorem.

For the case $p<\infty$ the argument is now based on the observation that $F' = F|L^\infty(T)$ satisfies the hypotheses for the case $p=\infty$ and therefore possesses a unique representing function ϕ' satisfying (a). Vitalli's convergence theorem then implies that $\phi = \phi'$ has the given properties. The converse utilizes a result of Krasnoselskii's on continuity of the transformation $x \rightarrow \phi \circ x$.

REMARK. In the linear case this reduces to the Riesz representation theorem, modulo a proof that $\phi(x, t) = xa(t)$ is in $\mathrm{Car}^p(T)$ if and only if $a \in L^q(T)$, $1 \leq p < \infty$.

THEOREM. *With* (T, Σ, μ) *as in the preceding theorem let* A *be a transformation such that* $A: L^p(T) \rightarrow C(S)$, $1 \leq p \leq \infty$, *where* S *is a compact Hausdorff space. Suppose* A *satisfies the conditions*

($\mathrm{i_A}$) *$A(x+y)=A(x)+A(y)$ when $xy=0$ a.e.,*

($\mathrm{ii_A}$) *A is uniformly continuous relative to L^∞ norm on each bounded subset of $L^\infty(T)$,*

($\mathrm{iii_A}$) *A is continuous relative to L^p norm, if $p<\infty$, and is continuous with respect to bounded a.e. convergence, if $p=\infty$.*

Then there exists a transformation Φ: $S\to\mathrm{Car}^p(T)$ such that

$$(*) \qquad A(x)(s) = \int_T \Phi(s) \circ x \, d\mu.$$

The transformation Φ can be taken to satisfy

(a) *$\Phi(s)\circ 0=0$ a.e. for all $s\in S$,*

in which case $\Phi(s)$ is unique, for each s, up to sets of the form $R\times N$ with N a null set in T. Moreover Φ has the following additional properties:

(b) *the mapping $s\to\Phi(s)\circ x\in L^1(T)$ is weakly continuous for each $x\in L^p(T)$,*

(c) *the mapping $x\to\Phi(s)\circ x$ is uniformly continuous relative to L^∞ norm on each bounded subset of $L^\infty(T)$, uniformly in s,*

(d) *the mapping $x\to\Phi(s)\circ x$ is weakly continuous, on $L^p(T)$, uniformly in s, if $p<\infty$; if $x_n\to x$ boundedly a.e. then $\lim_{\mu(E)\to 0}\int_E(\Phi(s)\circ x_n)d\mu\to 0$, uniformly in s and n, if $p=\infty$.*

Conversely, every transformation Φ: $S\to\mathrm{Car}^p(T)$, $1\leqq p\leqq\infty$, satisfying (a), (b), (c), (d) *determines by means of* (*) *a transformation $A: L^p(T)\to C(s)$ satisfying* ($\mathrm{i_A}$), ($\mathrm{ii_A}$) *and* ($\mathrm{iii_A}$).

The above result also extends to σ-finite measure spaces. For $p=\infty$ it is valid if the following condition is added:

(e) if $x_n\to x$ boundedly a.e., then for any sequence $E_j\downarrow\varnothing$, $\int_{E_j}(\Phi(s)\circ x_n)d\mu\to 0$, uniformly in s and n. For $p<\infty$ it is valid if the phrase 'bounded subset of $L^\infty(T)$' is replaced by 'bounded subset of $L^\infty(T)$ which is supported by a set of finite measure.'

The proof utilizes the preceding theorem on functionals together with the Vitalli-Hahn-Saks theorem on convergence of measures.

REMARK. For the linear case this result is well known (see [4, p. 490]).

I wish to thank my colleagues Charles Coffman and Kondagunta Sundaresan, the former for several stimulating conversations and the latter for a very interesting comment.

This research was partially supported by the National Science Foundation under Grant GP 7607.

REFERENCES

1. R. V. Chacon and N. Friedman, *Additive functionals*, Arch. Rational Mech. Anal. 18 (1965), 230–240.

2. B. D. Coleman and V. J. Mizel, *Norms and semi-groups in the theory of fading memory*, Arch. Rational Mech. Anal. **23** (1966), 87–123.

3. N. Friedman and M. Katz, *Additive functionals on L^p spaces*, Canad. J. Math. **18** (1966), 1264–1271.

4. N. Dunford and J. T. Schwartz, *Linear operators*. Part 1, Interscience, New York, 1958.

5. I. M. Gelfand and N. Ya Vilenkin, *Generalized functions*. Vol. 4, Academic. Press, New York, 1964, pp. 273–278.

6. M. A. Krasnosel'skii, *Topological methods in the theory of nonlinear integral equations*, translated by J. Burlak, Macmillan, New York, 1964, pp. 20–32.

7. A. D. Martin and V. J. Mizel, *A representation theorem for certain nonlinear functionals*, Arch. Rational Mech. Anal. **15** (1964), 353–367.

8. V. J. Mizel and K. Sundaresan, *Representation of additive and biadditive functionals*, Arch. Rational Mech. Anal. **30** (1968), 102–126.

9. K. Sundaresan, *Additive functonals on Orlicz spaces*, Studia Math. **32** (1968).

Carnegie-Mellon University, Pittsburgh, Pennsylvania 15213