CONVEXITY PROPERTIES OF NONLINEAR MAXIMAL MONOTONE OPERATORS

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Let X be a real Banach space with dual X^*. A *monotone operator* from X to X^* is by definition a (generally multivalued) mapping T such that

$$\langle x - y, x^* - y^* \rangle \geqq 0 \quad \text{whenever} \quad x^* \in T(x), \quad y^* \in T(y)$$

(where $\langle \cdot , \cdot \rangle$ denotes the pairing between X and X^*). Such an operator is said to be *maximal* if there is no monotone operator T' from X to X^*, other than T itself, such that $T'(x) \supset T(x)$ for every X. The *effective domain* $D(T)$ and *range* $R(T)$ of a monotone operator T are defined by

$$D(T) = \{x \mid T(x) \neq \emptyset\} \subset X,$$
$$R(T) = \bigcup \{T(x) \mid x \in X\} \subset X^*.$$

Minty [9] has shown that, when X is finite-dimensional and T is a maximal monotone operator, the sets $D(T)$ and $R(T)$ are *almost convex*, in the sense that each contains the relative interior of its convex hull. The purpose of this note is to announce some generalizations of Minty's result to infinite-dimensional spaces.

A subset C of X will be called *virtually convex* if, given any relatively (strongly) compact subset K of the convex hull of C and any $\epsilon > 0$, there exists a (strongly) continuous single-valued mapping ϕ from K into C such that $\|\phi(x) - x\| \leqq \epsilon$ for every $x \in K$. It can be shown that, in the finite-dimensional case, C is virtually convex if and only if C is almost convex, so that the following result contains Minty's result as a special case.

THEOREM 1. *Let X be reflexive, and let T be a maximal monotone operator from X to X^*. Then the strong closures of $D(T)$ and $R(T)$ are convex. If in addition X is separable, or if X is an L^p space with $1 < p < \infty$, $D(T)$ and $R(T)$ are virtually convex.*

The proof of Theorem 1, which will appear in [12], is made possible by recent results of Asplund [1], [2] concerning the existence of

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single-valued duality mappings $J: X \to X^*$, and results of Browder [5] concerning the invertibility of mappings of the form $T + \lambda J$, $\lambda > 0$, where T is a maximal monotone operator.

Since the subdifferential ∂f of a lower semicontinuous proper convex function f on X is a maximal monotone operator (Rockafellar [14]), Theorem 1 yields a new result about the existence of subgradients. (It has been observed elsewhere [3] that the strong closures of $D(\partial f)$ and $R(\partial f)$ are convex even when X is not reflexive.)

COROLLARY. *Let X be reflexive and separable (or an L^p space with $1 < p < \infty$), and let f be a lower semicontinuous proper convex function on X. Then $D(\partial f)$ and $R(\partial f)$ are virtually convex.*

Theorem 1 is applicable in particular to any single-valued monotone operator T with $D(T) = X$ such that T is hemicontinuous, i.e. continuous from line segments in X to the weak* topology of X^*, since such a T is known to be maximal (Browder [4]).

The following convexity result covers certain cases where X is not reflexive. Here T is said to be *locally bounded* at a point x if there exists a neighborhood U of x such that the set

$$T(U) = \bigcup \{T(u) \mid u \in U\}$$

is bounded in X^*.

THEOREM 2. *Let T be a maximal monotone operator from X to X^*. Suppose either that the convex hull of $D(T)$ has a nonempty interior, or that X is reflexive and there exists a point of $D(T)$ at which T is locally bounded. Then the interior of $D(T)$ is a convex set whose (strong) closure is the closure of $D(T)$. Moreover, T is locally bounded at every interior point of $D(T)$, whereas T is not locally bounded at any boundary point of $D(T)$.*

The local boundedness assertion of Theorem 2 strengthens a result of Kato [8], according to which a monotone operator T is locally bounded at any interior point of $D(T)$ where it is hemibounded.

Theorem 2 will be deduced in [13] from a more general theorem for locally convex spaces. The theorem of Debrunner-Flor [6] plays an important role in the proof.

The consequences of Theorem 2 include:

COROLLARY 1. *Let X be reflexive, and let T be a maximal monotone operator from X to X^* such that the convex hull of $R(T)$ has a nonempty interior. Then the interior of $R(T)$ is a convex set whose closure is the (strong) closure of $R(T)$.*

COROLLARY 2. *Let T be a maximal monotone operator from X to X^*, and let D_0 be the subset of $D(T)$ where T is single-valued. Then T is demicontinuous on D_0, i.e. continuous as a single-valued mapping from D_0 in the strong topology to X^* in the weak* topology.*

COROLLARY 3. *Let X be reflexive, and let T be a maximal monotone operator from X to X^*. Suppose there exists a subset B of X such that* 0 *belongs to the interior of the convex hull of*

$$T(B) = \bigcup \{T(x) \mid x \in B\}.$$

Then there exists an x such that $0 \in T(x)$.

COROLLARY 4. *Let X be reflexive, and let T be a maximal monotone operator from X to X^*. In order that $R(T)$ be all of X^*, it is necessary and sufficient that, whenever $x_i^* \in T(x_i)$ for $i = 1, 2, \cdots$, and $\|x_i\| \to \infty$, then the sequence $x_1^*, x_2^*, \cdots$, has no strongly convergent subsequence.*

Corollary 2 may be compared with the result of Kato [8] that a single-valued monotone operator T is demicontinuous on any open subset of $D(T)$ where it is hemicontinuous. Corollary 3 is a generalization of the main existence theorem of Minty [10], which requires in effect that 0 be an interior point of the convex hull $T_0(B)$, where T_0 is some mapping such that $T_0(x) \subset T(x)$ for every x and

$$\sup_{x \in B} \sup_{x^* \in T_0(x)} \langle x, x^* \rangle < \infty.$$

The necessary and sufficient condition in Corollary 4 is satisfied, in particular, when the following condition is satisfied: whenever

$$x_i^* \in T(x_i) \quad \text{for } i = 1, 2, \cdots,$$

and

$$\lim_{i \to \infty} \|x_i\| = \infty, \quad \text{then} \quad \lim_{i \to \infty} \|x_i^*\| = \infty.$$

(The two conditions are equivalent, of course, when X is finite-dimensional.) The sufficiency of the latter condition for $R(T)$ to be all of X^* has previously been established by Browder [4, Theorem 4].

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