A FACTOR THEOREM FOR FRÉCHET MANIFOLDS

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1. Introduction. A *Frechet manifold* (or *F-manifold*) is a separable metric space M having an open cover of sets each homeomorphic to an open subset of the countable infinite product of open intervals, s. A *Q-manifold* is a separable metric space M having an open cover of sets each homeomorphic to an open subset of the Hilbert cube, I^∞. It is known that all separable metric Banach manifolds modeled on separable infinite-dimensional Banach spaces are F-manifolds. The following are the principle theorems of this paper.

THEOREM I. *If M is any F-manifold, then $s\times M$ is homeomorphic to M.*

THEOREM II. *If M is any Q-manifold, then $I^\infty \times M$ is homeomorphic to M.*

Since s is known, [1] or [3], to be homeomorphic to $s\times I^\infty$, from Theorem I we immediately have the following.

COROLLARY. *If M is any F-manifold, then $I^\infty\times M$ is homeomorphic to M.*

Almost identical proofs of Theorems I and II can be given. To emphasize the ideas of our proofs of Theorems I and II we shall outline instead a proof of the similar but notationally easier

THEOREM I′. *If M is any F-manifold and J^0 is the open interval $(-1, 1)$, then $J^0\times M$ is homeomorphic to M.*

2. Lemma 2.1 implies Theorem I′.

DEFINITION. Let r be a map, i.e. continuous function, of a space X into the closed unit interval $[0, 1]$. Let $J^0(0)=\{0\}$ and for $t\in(0, 1]$, let $J^0(t)=(-t, t)$. Then $J^0 \times^r X=\{(y, x)\in J^0\times X: y\in J^0(r(x))\}$ is the *variable product* of J^0 by X (with respect to r).

LEMMA 2.1. *Let U be an open subset of s, let $V\subset W\subset U$ where W is open and V is closed in U, and let $J^0 \times^{r_0} U$ be a variable product of J^0*

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by U. There exists a homeomorphism H of $J^0 \times^{r_0} U$ onto a variable product $J^0 \times^r U$ such that (1) $r \leqq r_0$, (2) $r(V) = 0$, *and* (3) $H|J^0 \times^{r_0} [(U\setminus W)\cup r_0^{-1}(0)]$ *is the identity.*

Outline of a proof that Lemma 2.1 implies Theorem I′. Since M is separable and metric, there exists a countable star-finite open cover G of M with sets homeomorphic to open subsets of s. (By *star-finite* cover we mean a cover such that the closure of each element intersects only finitely many closures of other members of the cover.) Thus if $V \subset U$ where $U \in G$ and V is closed in M and $J^0 \times^{r_0} M$ is a variable product of J^0 by M, then by taking an open set W in M such that $W \supset V$ and $\mathrm{Cl}_M W \subset U$, Lemma 2.1 will imply the existence of a homeomorphism of $J^0 \times^{r_0} U$ that has an automatic extension to a homeomorphism H of $J^0 \times^{r_0} M$ where (1) $r \leqq r_0$, (2) $r(V)=0$, and (3) $H|J^0 \times^{r_0} [(M\setminus U)\cup r_0^{-1}(0)]$ is the identity.

As suggested by Theorem 2 of [2], we take a special ordering of the elements of G, say $\{U_i\}_{i>0}$. Now take a cover $\{V_i\}_{i>0}$ of M where, for each $i>0$, V_i is a closed set contained in U_i. For each $i>0$, let H_i be a homeomorphism from $J^0 \times^{r_{i-1}} M$ onto $J^0 \times^{r_i} M$ where $r_i \leqq r_{i-1} \leqq \cdots \leqq r_0 = 1$, $r_i(V_i)=0$, and $H_i|J^0 \times^{r_{i-1}} (M\setminus U_i)$ is the identity. Then $(H_i \circ \cdots \circ H_1)_{i>0}$ converges to a homeomorphism of $J^0 \times M$ onto $\{0\} \times M$ which is homeomorphic to M.

3. Two lemmas leading to Lemma 2.1. Let $s = \prod_{i>0} J_i^0$ where for each $i>0$, $J_i^0 = J^0$. Let $\pi: J^0 \times s \to s$ be the natural projection onto s and for $n>0$, let π_n be defined on s as follows. For $z=(z_1, z_2, \cdots) \in s$, let $\pi_n(z) = (z_1, \cdots, z_n, 0, 0, \cdots)$. Also, for Y a space and $f: s \to Y$, define $f^*: J^0 \times s \to Y$ by $f^* = f\pi$.

Lemma 3.1. *There exists a map*

$$h: (J^0 \times s) \times [0, 1] \times [1, \infty) \to J^0 \times s$$

such that if $t \in [0, 1]$ and $u \in [1, \infty)$ are fixed where $n \leqq u$, the map

$$H: J^0 \times s \to J^0 \times s$$

defined by $H(p) = h(p, t, u)$ for $p \in J^0 \times s$ is a homeomorphism of $J^0 \times s$ onto $J^0 \times^r s$ where (1) $r = 1 - t$, (2) *if $t=0$, H is the identity, and* (3) $\pi_n^* = \pi_n^* H$.

Outline of Proof. It suffices to describe h. We first describe for any integer $n>0$ a map h' of $(J^0 \times s) \times [0, 1) \times \{n\}$ onto $J^0 \times s$.

For $x = (x_0, x_1, \cdots) \in J^0 \times s$, let $h'(x, 0, n) = x$ and as t varies from 0 to $\frac{1}{2}$ let the 0th and $(n+1)$th coordinates of x be "rotated" so that at $t = \frac{1}{2}$, (x_0, x_{n+1}) becomes $(x_{n+1}, -x_0)$ while for $0 \leqq t \leqq \frac{1}{2}$ all other coordinates are left fixed.

Thus, $h'(x, \frac{1}{2}, n) = (x_{n+1}, x_1, \cdots, x_n, -x_0, x_{n+2}, \cdots)$. For $i>0$, as t varies from $1-2^{-i}$ to $1-2^{-i-1}$, "rotate" the 0th and $(n+i+1)$th coordinates leaving all other coordinates fixed so that for each $i>0$,

$$h'(x, 1 - 2^{-i}, n)$$
$$= (x_{n+i}, x_1, \cdots, x_n, -x_0, -x_{n+1}, \cdots, -x_{n+i-1}, x_{n+i+1}, \cdots).$$

To define h from h' we specify that $h(x, 1, n) = (0, x_1, \cdots, x_n, -x_0, -x_{n+1}, -x_{n+2}, \cdots)$ and we introduce for any time $t<1$, a factor of $1-t$ in the 0th coordinate place of h'. Indeed for each integer $n \geqq 1$, $h|(J^0 \times s) \times [0, 1] \times \{n\}$ becomes an isotopy. It is now possible to extend the domain of h for values of u between n and $n+1$ by use of "rotations" similar in nature to the "rotations" used in defining h' for a fixed n.

An open set E of s is an *n-basic* open set in s if $E = E_1 \times \cdots \times E_n \times \prod_{i>n} J_i^0$ where each E_i is open in J_i^0 and is a subinterval of J_i^0.

DEFINITION. Let W be open in s and let $\{G_i\}$ be a star finite collection of m_i-basic open sets in s whose union is W. For each $x \in W$ let

$$m_x = \text{minimum}\{m_i: x \in G_i\}.$$

Let Y be a space. A map f: $W \to Y$ is a *local product map* of W with respect to the G_i and m_i if $f(x) = f(\pi_{m_x}(x))$ for each $x \in W$. If, additionally, $Y = [1, \infty)$ and $f(x) \geqq m_x$ for each $x \in W$, then f is a *local product indicator map* of W with respect to the G_i and the m_i.

The strategy is to replace the t and u of Lemma 3.1 by local product maps. The following technical lemma (not proved here) asserts the existence of the proper type of local product maps.

LEMMA 3.2. *Let U be an open subset of s and let $V \subset W \subset U$ and $A \subset U$ where W is open and V and A are closed in U. There exist a countable star finite collection $\{G_i\}$ of m_i-basic open sets in s whose union is $W \backslash A$ and maps ϕ: $U \backslash A \to [0, 1]$ and g: $W \backslash A \to [1, \infty)$ such that* (1) $\phi(V \backslash A) = 1$, (2) $\phi((U \backslash W) \backslash A) = 0$, (3) $\phi | W \backslash A$ *and g are a local product map and a local product indicator map, respectively, of $W \backslash A$ with respect to the G_i and m_i, and* (4) *g is unbounded near A, that is, for any $x \in A \cap \mathrm{Cl}(W \backslash A)$ and $n>0$, there is a neighborhood $B(x)$ such that $g|(W \backslash A) \cap B(x) > n$.*

4. Proof of Lemma 2.1. By Lemma 3.2 take a star finite collection $\{G_i\}$ of m_i-basic open sets and the maps ϕ and g for the case when $A = r_0^{-1}(0)$. Now, let

$$h: (J^0 \times s) \times [0, 1] \times [1, \infty) \to J^0 \times s$$

be the map of Lemma 3.1 and define

$$H_1: J^0 \times W\backslash r_0^{-1}(0) \to J^0 \times s$$

by $H_1(p) = h(p, \phi^*(p), g^*(p))$ for $p \in J^0 \times W\backslash r_0^{-1}(0)$. It can be shown that H_1 is a homeomorphism onto $J^0 \times^{r_1} W\backslash r_0^{-1}(0)$ where $r_1 = 1 - \phi$. Clearly the map k from $J^0 \times^{r_0} W\backslash r_0^{-1}(0)$ to $J^0 \times W\backslash r_0^{-1}(0)$ defined by $k(y, z) = (yr_0^{-1}(z), z)$ is a homeomorphism. Also $k^{-1}H_1k$ is a homeomorphism from $J^0 \times^{r_0} W\backslash r_0^{-1}(0)$ onto $J^0 \times^{r} W\backslash r_0^{-1}(0)$ where $r = (1-\phi)r_0$. Now define $H: J^0 \times^{r_0} U \to J^0 \times^{r} U$ by $H = k^{-1}H_1k$ on $J^0 \times^{r_0} W\backslash r_0^{-1}(0)$ and $H =$ identity on $J^0 \times [(U\backslash W) \cup r_0^{-1}(0)]$. We show that H is continuous. Since $\phi((U\backslash W)\backslash r_0^{-1}(0)) = 0$, from condition 2 of Lemma 3.1 it follows that $H | J^0 \times (W\backslash r_0^{-1}(0))$ and the identity map on $J^0 \times [(U\backslash W)\backslash r_0^{-1}(0)]$ are compatible. To show that these are compatible with the identity $J^0 \times^{r_0} r_0^{-1}(0)$ we check the coordinatewise continuity of H. The continuity of r_0 gives the continuity of H on the first, or J^0, coordinate and g becoming unbounded near $r_0^{-1}(0)$ yields the continuity of H on the second, or U, coordinate. The other conditions of Lemma 2.1 are easily seen to be satisfied.

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