HOMOTOPY-EVERYTHING H-SPACES

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An *H-space* is a topological space X with basepoint e and a *multiplication* map $m: X^2 = X \times X \to X$ such that e is a homotopy identity element. (We take all maps and homotopies in the based sense. *We use k-topologies throughout* in order to avoid spurious topological difficulties. This gives function spaces a canonical topology.) We call X a *monoid* if m is associative and e is a strict identity.

In the literature there are many kinds of H-space: homotopy-associative, homotopy-commutative, A_∞-spaces [3], etc. In the last case part of the structure consists of higher *coherence* homotopies. In this note we introduce the concept of *homotopy-everything H-space* (*E-space* for short), in which all possible coherence conditions hold. We can also define E-maps (see §4). Our two main theorems are Theorem A, which classifies E-spaces, and Theorem C, which provides familiar examples such as BPL. Many of the results are folk theorems. Full details will appear elsewhere.

A space X is called an *infinite loop space* if there is a sequence of spaces X_n and homotopy equivalences $X_n \simeq \Omega X_{n+1}$ for $n \geqq 0$, such that $X = X_0$.

THEOREM A. *A CW-complex X admits an E-space structure with $\pi_0(X)$ a group if and only if it is an infinite loop space. Every E-space X has a "classifying space" BX, which is again an E-space.*

1. The machine. This constructs numerous E-spaces.

Consider the category $\mathcal{I}$ of real inner-product spaces of countable (algebraic) dimension and linear isometric maps between them. As examples we have $\mathbf{R}^\infty$ with orthonormal base $\{e_1, e_2, e_3, \cdots\}$, and its subspace $\mathbf{R}^n$ with base $\{e_1, e_2, \cdots, e_n\}$, which is all there are up to isomorphism. We topologize $\mathcal{I}(A, B)$, the set of all isometric linear maps from A to B, by first giving A and B the *finite* topology, which makes each the topological direct limit of its finite-dimensional subspaces.

LEMMA. *The space $\mathcal{I}(A, \mathbf{R}^\infty)$ is contractible.*

This is a consequence of two easily constructed homotopies:

(a) $i_1 \simeq i_2: A \to A \oplus A$,

(b) $i_1 \simeq u: \mathbf{R}^\infty \to \mathbf{R}^\infty \oplus \mathbf{R}^\infty$, for some *isomorphism* u.

Suppose we have a functor T defined on the category $\mathcal{I}$, taking topological spaces as values, and a continuous natural transformation $\omega: TA \times TB \to T(A \oplus B)$ called *Whitney sum*, such that:

(a) Tf is a continuous function of $f \in \mathcal{I}(A, B)$;
(b) $T\boldsymbol{R}^0$ consists of one point;
(c) ω preserves associativity, commutativity and units;
(d) $T\boldsymbol{R}^\infty$ is the direct limit of the spaces $T\boldsymbol{R}^n$.

THEOREM B. *$T\boldsymbol{R}^\infty$ is an E-space. If T happens to be monoid-valued, the classifying space* [2] *$BT\boldsymbol{R}^\infty$ agrees with that from Theorem A.*

As a (noncanonical) multiplication on $T\boldsymbol{R}^\infty$ we take

$$T\boldsymbol{R}^\infty \times T\boldsymbol{R}^\infty \xrightarrow[\omega]{} T(\boldsymbol{R}^\infty \oplus \boldsymbol{R}^\infty) \xrightarrow[Tf]{} T\boldsymbol{R}^\infty,$$

where $f: \boldsymbol{R}^\infty \oplus \boldsymbol{R}^\infty \to \boldsymbol{R}^\infty$ is any linear isometric embedding. The Lemma provides homotopy-associativity, since $f \circ (f \oplus 1) \simeq f \circ (1 \oplus f)$, homotopy-commutativity, and all higher coherence homotopies.

In the examples below we define TA explicitly only for finite-dimensional A, and note that axiom (d) extends the definition to the whole of $\mathcal{I}$. In each case the maps Tf and the Whitney sum ω are obvious, in view of the inner products.

EXAMPLES. 1. $TA = O(A)$, the orthogonal group of A. Then $T\boldsymbol{R}^\infty = O$.

2. $TA = U(A \otimes \boldsymbol{C})$, the unitary group of $A \otimes \boldsymbol{C}$. Then $T\boldsymbol{R}^\infty = U$.

3. $TA = BO(A)$, a suitable classifying space for $O(A)$, for example that given by [2]. Then $T\boldsymbol{R}^\infty = BO$.

4. $TA = F(A)$, the space of based homotopy equivalences of the sphere S^A, which is the one-point compactification $A \cup \infty$ of A, with ∞ as basepoint. Then $T\boldsymbol{R}^\infty = F$.

There is also a semisimplicial analogue, in which T takes semisimplicial values and $\mathcal{I}(A, B)$ is replaced by its singular complex.

5. $TA = \mathrm{Top}(A)$. A k-simplex of $\mathrm{Top}(A)$ is a fibre-preserving homeomorphism of $A \times \Delta^k$ over Δ^k, where Δ^k is the standard k-simplex. Then $T\boldsymbol{R}^\infty = \mathrm{Top}$.

6. The semisimplicial analogues of Examples 1–4.

7. The orientation-preserving versions of Examples 1–6.

8. $TA = PL(A)$, defined as $\mathrm{Top}(A)$ except taking only piecewise linear homeomorphisms of $A \times \Delta^k$. This *fails*. To cure this we need a new machine. Suffice it to say that as a k-simplex of $\mathcal{P}(A, B)$ we take a pair (ξ, f), where ξ is a p. l. subbundle of the product bundle $B \times \Delta^k$ over Δ^k, and $f: \xi \oplus (A \times \Delta^k) \cong B \times \Delta^k$ is a p. l. fibrewise homeomorphism that extends the inclusion of ξ.

THEOREM C. *We have the E-spaces O, SO, F, U, PL,* Top, *etc., their coset spaces F/PL,* $\Gamma =$ *"PL/O" etc., and all their iterated classifying spaces. The natural maps between these are all E-maps, including* $O \to PL$ *and* $PL \to \Gamma$.

2. Categories of operators. There are two variants: *with* or *without* permutations.

DEFINITION. In a *category of operators* $\mathcal{B}$

(a) the objects are $0, 1, 2, \cdots$;

(b) the morphisms from m to n form a topological space $\mathcal{B}(m, n)$, and composition is continuous;

(c) we are given a strictly associative continuous functor $\oplus : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ such that $m \oplus n = m + n$;

(d) if $\mathcal{B}$ has permutations, we are also given for each n a homomorphism $S_n \to \mathcal{B}(n, n)$, S_n the symmetric group on n letters. (We omit any symbol for this homomorphism.) In the case with permutations we impose two further axioms:

(i) if $\pi \in S_m$ and $\rho \in S_n$ then $\pi \oplus \rho$ lies in S_{m+n} and is the usual sum permutation;

(ii) given any r morphisms $\alpha_i : m_i \to n_i$ and $\pi \in S_r$, we have

$$\pi(\mathbf{n}) \circ (\alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_r) = \pi(\alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_r) \circ \pi(\mathbf{m}),$$

where $m = \Sigma m_i$, $n = \Sigma n_i$, π permutes the factors of $\alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_r$, and the permutation $\pi(\mathbf{n}) \in S_n$ is obtained from π by replacing i by a block of n_i elements. We require functors to preserve all this structure.

EXAMPLES. 1. End_X, for a based space X. $\mathrm{End}_X(m, n)$ is the space of all (based) maps $X^m \to X^n$, where X^n is the nth power of X. The functor $\oplus$ is just $\times$. This example has permutations.

DEFINITION. The category $\mathcal{B}$ of operators *acts on* X, or X *is a* $\mathcal{B}$*-space*, if we are given a functor $\mathcal{B} \to \mathrm{End}_X$.

2. $\mathcal{A}$. $\mathcal{A}(m, n)$ is the set of all order-preserving maps $\{1, 2, \cdots, m\} \to \{1, 2, \cdots, n\}$. Then an $\mathcal{A}$-space is a monoid.

3. $\mathcal{S}$. $\mathcal{S}(m, n)$ is the set of *all* maps $\{1, 2, \cdots, m\} \to \{1, 2, \cdots, n\}$, including permutations. Then an $\mathcal{S}$-space is an abelian monoid.

DEFINITION. We call X an *E-space* if we are given a category $\mathcal{B}$ of operators with permutations acting on X, for which $\mathcal{B}(n, 1)$ is contractible for all n. (We do not single out any canonical $\mathcal{B}$.)

4. $\mathcal{I}$. Define $\mathcal{I}(m, n) = \mathcal{I}((\mathbf{R}^\infty)^m, (\mathbf{R}^\infty)^n)$ as in §1. By the Lemma any $\mathcal{I}$-space, for instance $T\mathbf{R}^\infty$, is an E-space.

5. $\mathcal{Q}_n$, a category of operators on the nth loop space $X = \Omega^n Y$, the space of all maps $(I^n, \partial I^n) \to (Y, o)$, where I^n is the standard n-cube, ∂I^n its boundary, and o the basepoint of Y. A point $\alpha \in \mathcal{Q}_n(k, 1)$ is a

collection of k n-cubes I_i^n linearly embedded in I^n, with disjoint interiors, and with axes parallel to those of I^n. It acts on X as follows: given $(f_1, f_2, \cdots, f_k) \in X^k$, the map $\alpha(f_1, f_2, \cdots, f_k): I^n \rightarrow Y$ is given by f_i on each little cube I_i^n, and zero elsewhere. Similarly for $\mathcal{Q}_n(k, r)$. We topologize $\mathcal{Q}_n(k, 1)$ as a subspace of $\boldsymbol{R}^{2kn}$. We observe that $\mathcal{Q}_n(k, 1)$ is $(n-2)$-connected, so that as n tends to ∞, Theorem A becomes plausible.

We say the category $\mathcal{B}$ of operators, without permutations, is *in standard form* if every morphism $\alpha: m \rightarrow n$ has uniquely the form $\alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_n$, where $\alpha_i: m_i \rightarrow 1$. For categories with permutations the definition is more complicated. Of our examples, 2, 3 and 5 are in standard form, but 1 and 4 are not.

3. The bar construction. Suppose given a category $\mathcal{B}$ of operators, in standard form. We consider words $[\alpha_0 | \alpha_1 | \cdots | \alpha_k]$, where $k \geqq 0$ and each α_i is a morphism in $\mathcal{B}$ and $\alpha_0 \circ \alpha_1 \circ \cdots \circ \alpha_k$ exists.

DEFINITION. The category $W^0\mathcal{B}$ has as morphisms from m to n those words $[\alpha_0 | \alpha_1 | \cdots | \alpha_k]$ for which the composite exists and is in $\mathcal{B}(m, n)$, subject to the relations and their consequences:

$[\alpha \oplus \beta] = [\alpha \oplus 1 | 1 \oplus \beta] = [1 \oplus \beta | \alpha \oplus 1]$ for appropriate identities 1;

$[1]$ is an identity in $W^0\mathcal{B}$;

$[\alpha | \pi] = [\alpha \circ \pi]$ and $[\pi | \beta] = [\pi \circ \beta]$ if $\mathcal{B}$ has permutations π. Composition in $W^0\mathcal{B}$ is by juxtaposition.

To form the category $W\mathcal{B}$, we take for each morphism x in $W^0\mathcal{B}$ a cube $C(x)$ of suitable dimension, having x as vertex, and identify the faces not containing x with certain cubes $C(x_i)$ of lower dimension, where x_i runs through the words formed from x by one "amalgamation." The categories $W^0\mathcal{B}$ and $W\mathcal{B}$ inherit obvious identification topologies. For composition we have $C(x) \circ C(y) \subset C(x \circ y)$ as a face containing $x \circ y$, and $\oplus: C(x) \times C(y) \cong C(x \oplus y)$. The *augmentation* $\epsilon: W\mathcal{B} \rightarrow \mathcal{B}$ is defined by $\epsilon[\alpha_0 | \alpha_1 | \cdots | \alpha_k] = \alpha_0 \circ \alpha_1 \circ \cdots \circ \alpha_k$ and $\epsilon C(x) = \epsilon x$.

A $W\mathcal{A}$-space, with $\mathcal{A}$ as in §2, is approximately an A_∞-space [3]. In particular, the familiar pentagon in $W\mathcal{A}(4, 1)$ is now subdivided into five squares.

For the following theorems we need a slightly different category $\mathcal{B}'$ augmented over $\mathcal{B}$, in which $\mathcal{B}'(1, 1)$ differs from $\mathcal{B}(1, 1)$ by a whisker. However, we may replace $\mathcal{B}'$ by $\mathcal{B}$ in the theorems whenever the identity 1 in $\mathcal{B}(1, 1)$ is an isolated point.

We call an augmentation functor $\theta: \mathcal{C} \rightarrow \mathcal{B}$ *fibre-homotopically trivial* if for each n there exists a section $\chi: \mathcal{B}(n, 1) \rightarrow \mathcal{C}(n, 1)$ such that $\chi \circ \theta$ is fibrewise homotopic to the identity map of $\mathcal{C}(n, 1)$, S_n-equivariantly if $\mathcal{B}$ and $\mathcal{C}$ have permutations.

THEOREM D. (a) ϵ: *$W\mathcal{B}' \to \mathcal{B}' \to \mathcal{B}$ is fibre-homotopically trivial.*

(b) *Given any category of operators $\mathcal{C}$ augmented over $\mathcal{B}$ by a fibre-homotopically trivial functor, there exists a functor $W\mathcal{B}' \to \mathcal{C}$ that lifts $W\mathcal{B}' \to \mathcal{B}$. (It is not unique.)*

THEOREM E. *Suppose X and Y have the same homotopy type, and $W\mathcal{B}'$ acts on X. Then we can make $W\mathcal{B}'$ act on Y.*

4. Maps between H-spaces. Suppose X and Y are $W\mathcal{B}$-spaces and $f: X \to Y$ a map. We call f a *$W\mathcal{B}$-homomorphism* if it commutes with the action of $W\mathcal{B}$. We need a weaker homotopy notion.

Let $\mathcal{L}_n$ be the "linear" category with objects $0, 1, 2, \cdots, n$ and one morphism $i \to j$ whenever $i \leqq j$. We can generalize the bar construction in §3 to form $W(\mathcal{B} \times \mathcal{L}_n)$, a category which we make act on $(n+1)$-tuples of spaces. (Note that in $\mathcal{B} \times \mathcal{L}_n$ $\oplus$ is not quite a functor, because it is not everywhere defined.) We define a *homotopy $\mathcal{B}$-map* from X to Y as an action of $W(\mathcal{B} \times \mathcal{L}_1)$ on the pair (X, Y), that induces the given $W\mathcal{B}$-structures on X and Y. Similarly, an action on E-spaces X and Y of a suitable category $\mathcal{C}$ such that $\mathcal{C}(X^n, Y)$ is contractible for all n is called an *E-map.*

THEOREM F. *Let X and Y be $W\mathcal{B}'$-spaces, and $f: X \to Y$ a homotopy $\mathcal{B}'$-map that is also a homotopy equivalence. Then any homotopy inverse $g: Y \to X$ admits the structure of homotopy $\mathcal{B}'$-map.*

We cannot form the category of $W\mathcal{B}$-spaces and homotopy $\mathcal{B}$-maps, because unless one of them is a $W\mathcal{B}$-homomorphism, the composite of two homotopy $\mathcal{B}$-maps is defined only up to a homotopy, which is defined only up to a homotopy, which is . . . . Instead we form a semi-simplicial complex K, whose n-simplexes are actions of $W(\mathcal{B} \times \mathcal{L}_n)$ on $(n+1)$-tuples of spaces; in particular its vertices are $W\mathcal{B}$-spaces and its edges are homotopy $\mathcal{B}$-maps.

THEOREM G. *This complex K satisfies the "restricted Kan extension condition," in which the omitted face is not allowed to be the first or the last.*

This result provides everything we need for composition up to homotopy etc., and allows the formation of the category of $W\mathcal{B}$-spaces and homotopy classes of homotopy $\mathcal{B}$-maps.

5. Structure theory. The following theorem is essentially due to Adams.

THEOREM H. *Given a $W\mathcal{A}$-space X, there is a universal monoid MX equipped with a homotopy $\mathcal{A}$-map $i: X \to MX$, such that any homotopy*

$\mathfrak{A}$-map $f: X \to G$ to a monoid G factors uniquely as $g \circ i$ with g a monoid homomorphism. Moreover, if X is a CW-complex the map i is a homotopy equivalence.

Our main technical result for proving Theorem A is:

THEOREM J. *Let X be an E-space, so that in particular it admits a $W\mathfrak{A}$-structure. Then the classifying space* [2] *BMX is also an E-space.*

For Theorem A we then *define* $BX = BMX$.

6. Cohomology theories. Given an E-space Y such that Y is a CW-complex and $\pi_0(Y)$ is a group, we define [1] a graded additive cohomology theory on CW-pairs by setting

$$t^n(X, A) = [X/A, B^nY],\ t^{-n}(X, A) = [X/A, \Omega^nY] \qquad \text{for } n \geqq 0,$$

whose coefficient groups vanish for $n > 0$. Let us call a cohomology theory with this property *connective.*

THEOREM K. *Every connective graded additive cohomology theory on CW-pairs arises from some such E-space Y, which is uniquely determined up to homotopy equivalence of E-spaces.*

In particular the E-space $\mathbf{Z}\times BU$ gives rise to the connective K-theory cK. This is more usually obtained by appealing to Bott periodicity and killing off the unwanted coefficient groups. In other cases we cannot appeal to Bott periodicity:

DEFINITION. We define *connective piecewise linear K-theory cK_{PL}* by using the E-space $\mathbf{Z}\times BPL$: for $n > 0$ we set

$$cK^n_{PL}(X, A) = [X/A, B^n(\mathbf{Z} \times BPL)].$$

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