TOPOLOGICAL EMBEDDINGS IN CODIMENSION ONE[1]

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Communicated by O. G. Harrold, Jr. April 10, 1968

1. **Introduction.** Suppose Q^{n+1} is a piecewise linear $(n+1)$-manifold and M^n is a closed topological n-manifold embedded in int Q^{n+1}. We seek conditions on the embedding of M which insure that M has arbitrarily small neighborhoods which look like regular neighborhoods of a piecewise linear (PL) submanifold of Q. In particular, we would like M to be contained in a compact $(n+1)$-dimensional PL submanifold N of Q such that

(1) $M \subset \text{int } N$,

(2) M is a strong deformation retract of N, and

(3) $N - M$ is PL homeomorphic to bd $N \times [0, 1)$.

We call any compact (connected) PL submanifold N of Q satisfying (1) a PL *manifold neighborhood of M*.

We say that *$Q-M$ is 1-lc at M* if for each open set U containing M there is an open set V, $M \subset V \subset U$, such that each loop in $V-M$ is null homotopic in $U-M$. The purpose of this note is to show that, if M is simply connected and $n \geqq 5$, then M has PL manifold neighborhoods satisfying (2) and (3) above if and only if $Q-M$ is 1-*lc* at M.

All homology and cohomology groups will be singular with Z coefficients. i_* (i^*) will denote an inclusion induced map between homology or homotopy (cohomology) groups. The symbol $\approx$ means is isomorphic to or is PL homeomorphic to, depending on the context. I denotes the unit interval $[0, 1]$.

2. **Statement of results.** Let Q^{n+1} be a connected PL $(n+1)$-manifold, M^n a closed, 1-connected topological n-manifold embedded in int Q. Our main result is

THEOREM 1. *If $n \geqq 5$, there is a closed PL n-manifold M^* such that M has arbitrarily small PL manifold neighborhoods which are PL homeomorphic to $M^* \times I$ and satisfy* (2) *and* (3) *above if and only if $Q-M$ is* 1-*lc at M.*

The proof is postponed until §3.

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[1] The work in this paper overlaps the author's doctoral dissertation, which was written under the direction of Professor C. H. Edwards, Jr. at the University of Georgia.

[2] Partially supported by a National Science Foundation Graduate Fellowship.

Now if M is a PL submanifold of Q, M is locally flat, and thus $Q-M$ is 1-*lc* at M. It follows that the boundary components of any regular neighborhood of M are simply connected, so Theorem 1 and the h-cobordism theorem [4] (or Lemma 1 below) easily imply

COROLLARY 1. *If M^n, $n \geqq 5$, is a closed, 1-connected PL submanifold in the interior of a PL $(n+1)$-manifold, regular neighborhoods of M are PL homeomorphic to the product of some closed PL n-manifold with I.*

Corollary 1 generalizes a theorem of Husch, [3].

COROLLARY 2. *If M^n is a closed, 1-connected, topological n-manifold, $n \geqq 5$, which can be embedded in the interior of a PL manifold Q^{n+1} in such a way that $Q-M$ is 1-lc at M, then M has the homotopy type of a closed PL n-manifold.*

3. **Proof of Theorem 1.** We will need the following two lemmas.

LEMMA 1. *If W^{n+1} is a compact PL $(n+1)$-manifold, $n \geqq 5$, with exactly two boundary components A and B, both 1-connected, and W has the homotopy type of a closed, 1-connected n-manifold, then $W \approx A \times I$.*

PROOF. By the h-cobordism theorem, it suffices to show that the inclusion $A \subset W$ is a homotopy equivalence, and for this it is sufficient, by a theorem of Whitehead [7], to show that $H_q(W, A) \approx 0$ for all q. Since W is 1-connected, $H^1(W, B) \approx 0$, so by Poincaré duality, $H_n(W, A) \approx 0$. Since $H_n(A) \approx H_n(W) \approx Z$, the exact homology sequence of (W, A) shows that $i_*: H_n(A) \rightarrow H_n(W)$ is an isomorphism, so it follows from [6] that for each q, $i_*: H_q(A) \rightarrow H_q(W)$ is onto and $i^*: H^q(W) \rightarrow H^q(A)$ is 1-1. The exact homology and cohomology sequences of (W, A) show that for each q,

$$H_{q+1}(W, A) \approx \ker \{i_*: H_q(A) \rightarrow H_q(W)\},$$

and

$$H^{q+1}(W, A) \approx \operatorname{coker} \{i^*: H^q(W) \rightarrow H^q(A)\}.$$

By Poincaré duality for kernels [1], [6], there is an isomorphism ρ_q: $H_{q+1}(W, A) \rightarrow H^{n+1-q}(W, A)$ for each q. Composing ρ_q with the Poincaré duality isomorphism for (W, A, B) gives an isomorphism $H_{q+1}(W, A) \approx H^{n+1-q}(W, A) \approx H_q(W, B)$ and a similar argument gives $H_{q+1}(W, B) \approx H_q(W, A)$.

Since $H_0(W, A) \approx H_0(W, B) \approx 0$, it follows that $H_q(W, A) \approx 0$ for each q, and the proof is complete.

LEMMA 2. *If M^n, Q^{n+1} are as in Theorem 1 and $Q-M$ is 1-lc at M, then M has arbitrarily small PL manifold neighborhoods W such that*

(a) *W has exactly two boundary components, each* 1-*connected,*
(b) *W is* 1-*connected, and*
(c) *$W-M$ has two components, each* 1-*connected.*

PROOF. We may assume that Q is open and 1-connected, for if $\tilde{Q}\xrightarrow{p}\operatorname{int} Q$ is the (PL) universal cover of $\operatorname{int} Q$, the fact that M is 1-connected implies that p is a homeomorphism on each component of $p^{-1}(M)$. If $\tilde{M}$ is a component of $p^{-1}(M)$, a standard compactness argument shows that p is a homeomorphism on some neighborhood of $\tilde{M}$, so we may replace Q and M by $\tilde{Q}$ and $\tilde{M}$.

Now Alexander duality [5, Theorem 6.2.17] shows that $H_1(Q, Q-M)\approx Z$, and the reduced homology sequence of $(Q, Q-M)$ shows that $Q-M$ has exactly two components, say Q_1 and Q_2. Furthermore, it is easy to see that if U is any connected open neighborhood of M, $U-M$ has exactly two components, $Q_i\cap U$, $i=1, 2$.

Let U be any open neighborhood of M, and let V be a neighborhood of M such that $V\subset U$ and each loop in $V-M$ is null homotopic in $U-M$. We may also assume that each loop in $U-M$ is null homotopic in $Q-M$. Let W_1 be any PL manifold neighborhood of M in V, and suppose W_1 has more than one boundary component in Q_1. If A and B are two of these components, we may join them with a polyhedral arc α which lies, except for its endpoints, in $\operatorname{int} W_1-M$. If T is a small regular neighborhood of α in W_1-M, $\operatorname{Cl}(W_1-T)$ is a PL manifold neighborhood of M in V with one less boundary component in Q_1. Continuing this process on both sides of M, we can find a PL manifold neighborhood W_2 of M in V which has exactly one boundary component in each of Q_1 and Q_2. Since each loop in bd W_2 is null homotopic in $U-M$ by our choice of V, and since $n\geqq 5$, we may alter W_2 by "exchanging disks" [1] to get a PL manifold neighborhood W of M in U which satisfies (a). By two applications of the Van Kampen theorem, W is 1-connected, so (b) holds. $W-M$ has two components, and if N_j is the component contained in Q_j, $j=1, 2$, our assumption on U implies that $i_*: \pi_1(N_j)\rightarrow\pi_1(Q_j)$ is trivial, so the Van Kampen theorem implies that N_j is 1-connected, and the proof is complete.

PROOF OF THEOREM 1. Necessity is obvious. For the converse, let W be one of the PL manifold neighborhoods guaranteed by Lemma 2, small enough that M is a retract of W. Let V be the double of W and think of V as $W_1\cup W_2$ where the W_i are the two copies of W and $M\subset W_1$. Then $V-M$ is an open PL manifold, and Lemma 2 shows that $V-M$ has two simply connected ends. Alexander duality and the exact homology sequence of $(V, V-M)$ show that the homology of $V-M$ is finitely generated, so by the Browder, Levine, and Livesay boundary theorem [2], $V-M$ is PL homeomorphic to the interior of a compact PL manifold with two simply connected boundary components. This means that there is a compact, PL submanifold X^{n+1} of $V-M$ such that X has two simply connected boundary components, $W_2 \subset \operatorname{int} X$, and $(V-M)-\operatorname{int} X \approx \operatorname{bd} X \times [0, 1)$. Let $N=V-\operatorname{int} X$. Then N clearly satisfies conditions (1) and (3) of the introduction. Since V is orientable, Alexander duality gives $H^q(N, M) \approx H_{n+1-q}(V-M, \operatorname{int} X) \approx 0$ for each q. By the Universal Coefficient Theorem, $H_q(N, M) \approx 0$ for each q. N is simply connected by the arguments of Lemma 2, so by the Whitehead theorem, $i: M \to N$ is a homotopy equivalence. Since N retracts onto M, it follows that M is a strong deformation retract of N, so N satisfies (3). N is a product by Lemma 1, so the proof is complete.

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