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ON EIGENVALUE DISTRIBUTIONS FOR ELLIPTIC OPERATORS WITHOUT SMOOTH COEFFICIENTS. II

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In two previous papers the known asymptotic formula for the eigenvalues of a selfadjoint elliptic boundary value problem was extended to some cases of operators without smooth coefficients: to the Dirichlet problem in [1] and to the general coercive differential boundary value problem in [2]. The object of this note is to complete this study by proving the formula for general (i.e. not necessarily differential) boundary value problems on domains without smooth boundary. We use the methods of [1], [2]. It should be noted that the case of differential boundary conditions can be handled in a different way; see [3].

Let Ω be a bounded open set in R^n with boundary $\partial\Omega$ which is regular in the sense of Calderón [4], i.e. satisfying the "restricted cone condition." Let $\boldsymbol{A}=\sum a_\alpha(x)D^\alpha$ be an operator defined on Ω, with coefficients in $L^\infty(\Omega)$ and top-order coefficients uniformly continuous on Ω. We assume that $\boldsymbol{A}$ is formally selfadjoint and uniformly elliptic of order m. Let A be a selfadjoint realization of $\boldsymbol{A}$ in $L^2(\Omega)$, with domain $D(A)\subset H^m(\Omega)$. Set

$$c(A) = (2\pi)^{-n}\int_\Omega\int_{a(x,\xi)<1} d\xi dx, \tag{1}$$

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where $a(x, \xi)$ is the characteristic polynomial of $\boldsymbol{A}$ at x.

THEOREM. *For $\lambda > 0$, let $N(\lambda)$ be the number of eigenvalues of A with absolute value $<\lambda$, counting multiplicity. Then*

$$N(\lambda) \sim c(\boldsymbol{A})\lambda^{n/m} \qquad \text{as } \lambda \to \infty. \tag{2}$$

We may replace A by $A+tI$ for some real t and assume that A has a bounded inverse S. Set

$$\mu_j(S) = \inf_{\operatorname{codim} K = j-1} \ \sup_{u \in K, \|u\|=1} \|Su\|, \tag{3}$$

$$\alpha(S) = \liminf_{j \to \infty} j^{-m/n} \mu_j(S), \tag{4}$$

$$\beta(S) = \limsup_{j \to \infty} j^{-m/n} \mu_j(S). \tag{5}$$

The assertion of the Theorem is equivalent to $\alpha(S) = \beta(S) = d(\boldsymbol{A})$, where

$$d(\boldsymbol{A}) = (c(\boldsymbol{A}))^{m/n}. \tag{6}$$

By Theorem 5.2 of [2], $\alpha(S) \geqq d(\boldsymbol{A})$. We must show $\beta(S) \leqq d(\boldsymbol{A})$.

Let $\boldsymbol{A}_\epsilon$, $0<\epsilon<1$, be a family of formally selfadjoint operators of order m defined on a neighborhood of $\Omega \cup \partial\Omega$ and having C^∞ coefficients, such that the coefficients of the principal parts converge uniformly on Ω to those of the principal part of $\boldsymbol{A}$ as $\epsilon \to 0$. Let A_ϵ be the restriction of $\boldsymbol{A}_\epsilon$ to $D(A)$. For small ϵ, $\boldsymbol{A}_\epsilon$ is uniformly elliptic on Ω and A_ϵ has a bounded inverse S_ϵ. By Theorems 1.5 and 2.3 of [2], $\beta(S_\epsilon) \to \beta(S)$ as $\epsilon \to 0$. Clearly $d(\boldsymbol{A}_\epsilon) \to d(\boldsymbol{A})$, so it suffices to prove the following.

LEMMA. *Let $\boldsymbol{A}_0$ be a formally selfadjoint operator defined on a neighborhood of $\Omega \cup \partial\Omega$, having C^∞ coefficients, and uniformly elliptic on Ω, of order m. Let $\boldsymbol{A}_0$ be the restriction of $\boldsymbol{A}_0$ to a domain $D(A_0) \subset H^m(\Omega)$. Suppose A_0 has a bounded inverse S_0. Then $\beta(S_0) \leqq d(\boldsymbol{A}_0)$.*

PROOF. Let Ω_1 be a bounded neighborhood of $\Omega \cup \partial\Omega$ having a smooth boundary, such that $\boldsymbol{A}_0$ is defined and uniformly elliptic on Ω_1. Let A_1 be the Dirichlet realization of $\boldsymbol{A}_0$ in $L^2(\Omega_1)$. Then A_1 is selfadjoint and (2) holds for the eigenvalues of A_1. We may assume A_1 has a bounded inverse S_1.

Let $R: L^2(\Omega_1) \to L^2(\Omega)$ be the restriction mapping and $J = R^*: L^2(\Omega) \to L^2(\Omega_1)$ the extension mapping. Then $JR = P$, the projection of $L^2(\Omega_1)$ onto functions vanishing outside Ω. Let $H_1 = \{u \mid RS_1 u \in D(A_0)\}$.

This is a closed subspace of $L^2(\Omega_1)$; let P_1 be the corresponding projection. Then $S_0RP_1 = RS_1P_1$. If $v \in D(A_0) \subset H^m(\Omega)$, it can be extended [4] to a $u \in H_c^m(\Omega_1) \subset D(A_1)$. Therefore R maps H_1 onto $L^2(\Omega)$. Then J is a unitary mapping of $L^2(\Omega)$ onto $PH_1 = PL^2(\Omega_1)$ and R is its inverse; S_0 corresponds to JS_0R acting on $PH_1 = PP_1L^2(\Omega_1)$. Therefore

$$\mu_j(S_0) \leqq \mu_j(JS_0RPP_1) = \mu_j(JS_0RP_1)$$
$$= \mu_j(JRS_1P_1) = \mu_j(PS_1P_1) \leqq \mu_j(S),$$

and $\beta(S_0) \leqq \beta(S_1) = d(\boldsymbol{A}_1)$, where $\boldsymbol{A}_1$ denotes $\boldsymbol{A}_0$ considered as defined on Ω_1 rather than on Ω. We get the desired conclusion by shrinking Ω_1.

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