# EXTENDING CERTAIN TRANSFORMATION GROUP ACTIONS IN SEPARABLE, INFINITE-DIMENSIONAL FRECHET SPACES AND THE HILBERT CUBE

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A theorem of V. L. Klee, Jr. [9] asserts that any homeomorphism between two compact sets in a separable, infinite-dimensional Hilbert space can be extended to a homeomorphism of the entire space onto itself. Since it has recently been shown ([1], [5] and [6]) that all separable, infinite-dimensional Frechet spaces are homeomorphic (a Frechet space being a metrizable, complete, locally convex linear topological space), this result holds in all such spaces. A salient property of compact sets in such spaces is that for each nonnull, homotopically trivial open set $U$, the relative complement in $U$ of a compact set is also nonnull and homotopically trivial (see [3, §§3–5]). This property has been called "property $Z$" by R. D. Anderson, who has extended Klee's theorem above to the following in [3].

THEOREM A (ANDERSON). *If $X$ is a separable, infinite-dimensional Frechet space or the Hilbert cube, then any homeomorphism between two closed subsets of $X$ which have property $Z$ can be extended to a homeomorphism of $X$ onto itself.*

It might be noted here that any compact metric space can be imbedded in the Hilbert cube as a set with property $Z$ and that any separable, complete, metric space can be imbedded in any separable, infinite-dimensional Frechet space as a closed subset with property $Z$ (see [3]).

If $X$ is a separable, infinite-dimensional Frechet space or the Hilbert cube, and if $Y$ is a closed subset of $X$ with property $Z$, then Theorem A provides a function $F$ from $H(Y)$, the group of homeomorphisms of $Y$ onto itself, into $H(X)$ such that for $h$ in $H(Y)$, $F(h)|Y=h$. This function is not, in general, a homomorphism, for the extension procedure does not commute with composition. It is, however, in many cases possible to construct such a homomorphism and to make it continuous in the sense of transformation groups through a linearization procedure suggested by work of P. C. Baayen and J. de Groot [4]. That is the purpose of this paper. (A transformation group is a triple $(G, X, v)$, where $G$ is a topological group, $X$ is a topological space, and $v$ is a continuous function from $G\times X$ into $X$ such that

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(1) for $g_1$ and $g_2$ in $G$, $v(g_1g_2, x)=v(g_1, v(g_2, x))$ for each $x$ in $X$, and (2) if $e$ is the identity of $G$, then for each $x$ in $X$, $v(e, x)=x$. The function $v$ is called the action of $G$ on $X$ as a transformation group.) The first theorem concerns the Hilbert cube.

THEOREM 1. *Any transformation group action of a topological group $G$ on a closed subset of the Hilbert cube with property Z can be extended to an action of $G$ on the Hilbert cube as a transformation group.*

PROOF. Let $(G, X, v)$ be the given transformation group, and let $H(X)$ be the group of all homeomorphisms of $X$ onto itself. Under the topology of uniform convergence, $H(X)$ is a separable, metrizable, topological group, and the evaluation function is an action of $H(X)$ on $X$ as a transformation group.

Let $l_2$ denote the real Hilbert space of all square-summable sequences of real numbers with the norm $\|(x_i)\| = (\sum_{i=1}^{\infty} x_i^2)^{1/2}$, and let $E$ be the space of all continuous functions from $H(X)$ into $l_2$ with the topology of uniform convergence on compacta. Now $E$ is a Hausdorff, locally convex, linear topological space, and because $H(X)$ is metrizable, $E$ is complete [**8**, p. 231]. Let $f_1$ be an imbedding of $X$ in $l_2$, and let $X'$ be the union of $f_1(X)$ with an infinite-dimensional compact set in $l_2$ disjoint from it. Let $f_2$: $X' \to E$ be the function defined by

$$(f_2(x'))(h) = \begin{cases} f_1hf_1^{-1}(x'), & \text{if } x' \in f_1(X) \\ x', & \text{if } x' \notin f_1(X) \end{cases}$$

for each $h$ in $H(X)$. It is not difficult to see that $f_2$ is an imbedding of $X'$ in $E$.

Let $K$ be the closed, convex hull of $f_2(X')$ in $E$, and let $L$ be the homomorphism of $H(X)$ into the group of linear homeomorphisms of $E$ onto itself defined by $(L(g)(z))(h)=z(hg)$, for all $z$ in $E$ and all $g$ and $h$ in $H(X)$. For each $g$ in $H(X)$, $L(g)$ carries $K$ into itself.

Let $w$: $H(X)\times K \to K$ be the function defined by $L$, that is, $w(g, z) =(L(g))(z)$ for all $z$ in $K$ and $g$ in $H(X)$. By the construction, $w(g, f_2, f_1(x)) =f_2f_1g(x)$ for each $g$ in $H(X)$ and $x$ in $X$. Further, $w$ is an action of $H(X)$ on $K$ as a transformation group. Since the closed, convex hull of a compact set in a Hausdorff, complete, locally convex, linear, topological space is compact [**10**, p. 60], $K$ is an infinite-dimensional, compact, convex set in such a space.

By a theorem of Klee [**9**], in order to show that $K$ is homeomorphic to the Hilbert cube, it suffices to find a countable family of continuous linear functionals on $E$ which separates points of $K$ (for then using the functionals, one may imbed $K$ linearly in $l_2$ and use a theorem of O.-H. Keller [**7**]). Such a family $P$ may be constructed on $E$ as follows: Let $\{h_m\}_{m>0}$ be a countable, dense set in $H(X)$, and let

$\{k_n\}_{n>0}$ be the family of projection functionals of $l_2$ onto its coordinate axes. Let $P=\{p_{mn}\}_{m,n>0}$, where for each $z$ in $E$, $p_{mn}(z)=k_n(z(h_m))$. Let the Hilbert cube $Y$ be represented as the countably infinite Cartesian product of the closed interval $[-1, 1]$ with itself, and let $q_n$ denote the projection of $Y$ onto its $n$th coordinate space. Let $K'=K\times[-1, 1]$, and let $w'$ be the action of $H(X)$ on $K'$ as a transformation group defined by $w'(g, (z, t))=(w(g, z), t)$, for $g$ in $H(X)$ and $(z, t)$ in $K'$. Let $f_3$ be a homeomorphism of $K$ onto $Y$, and let $f_4$ be the homeomorphism of $K'$ onto $Y$ defined by $f_4(z, t)=(t, q_1f_3(z), q_2f_3(z), \cdots)$, for $(z, t)$ in $K'$.

The function $f'$: $X\rightarrow Y$ defined by $f'(x)=f_4(f_2f_1(x), 1)$ is an imbedding of $X$ in $Y$ as a set with property $Z$, so by Theorem A there exists a homeomorphism $f''$ of $Y$ onto itself agreeing with $f'$ on $X$.

If $p$: $G\rightarrow H(X)$ is the homomorphism induced by $v$, that is, $p(g)(x)=v(g, x)$ for each $g$ in $G$ and $x$ in $X$, then it is easy to see that $p$ is continuous. Define $v'$: $G\times Y\rightarrow Y$ by $v'(g, y)=f''^{-1}f_4w'(p(g), f_4^{-1}f''(y))$. Now $v'$ is the desired action of $G$ on $Y$ extending $v$.

THEOREM 2. *Any transformation group action of a topological group $G$ on a closed, locally compact subset of a separable, infinite-dimensional Frechet space $Y$ can be extended to an action of $G$ on $Y$ as a transformation group.*

PROOF. Let $(G, X, v)$ be the given transformation group; let $X'$ be the one-point compactification of $X$, and let $f_1$ be the natural imbedding of $X$ in $X'$. Let $v'$: $G\times X'\rightarrow X'$ be the transformation group action of $G$ on $X'$ defined by

$$v'(g, x') = \begin{cases} f_1v(g, f_1^{-1}(x')), & \text{if } x' \in f_1(X') \\ x', & \text{if } x' \notin f_1(X') \end{cases}$$

for $g$ in $G$ and $x'$ in $X'$. Because $X'$ is compact and metrizable, there is an imbedding $f_2$ of $X'$ in the Hilbert cube $Q$ as a set with property $Z$.

Let $v''$ be the action of $G$ on $f_2(X')$ induced from $v'$ by $f_2$. By Theorem 1, there is an extension $w$ of $v''$ to an action of $G$ on $Q$. Note that for each $g$ in $G$, $w(g, f_2(X')-f_2f_1(X))-f_2(X')=f_2f_1(X)$. Let $Y'=Q\times s$, where $s$ is the countably infinite Cartesian product of the real line with itself. Now $Y'$ is homeomorphic to $s$ [2, Corollary 9.4], which is a separable, infinite-dimensional Frechet space and is hence homeomorphic to $Y$. Let $w'$ be the action of $G$ on $Y'$ as a transformation group defined by $w'(g, (y, z))=(w(g, y), z)$, for $g$ in $G$ and $(y, z)$ in $Y'$. Let $f_3$ be the imbedding of $X'$ in $Y'$ carrying $x'$ to $(f_2(x'), 0)$, where 0 is the origin of $s$, and denote by $x_0$ the point $f_3(X'-f_1(X))$.

By a theorem of Anderson [2, Corollary 5.5], $s$ is homeomorphic to the complement of any of its compacta, so there is a homeomorphism $f_4$ of $Y'-x_0$ onto $Y$. Let $w''$ be the action of $G$ on $Y$ as a transformation group induced from $w'$ by $f_4$. Now $f_4 f_3 f_1$ is an imbedding of $X$ in $Y$ as a closed set and $w''$ extends the action of $G$ on $f_4 f_3 f_1(X)$ induced from $v$, so in order to complete the proof it suffices to show that $X$ and $f_4 f_3 f_1(X)$ have property $Z$ in $Y$ and to apply Theorem A. However, any closed, locally compact subset of a separable, infinite-dimensional Frechet space has property $Z$, and one way to see this is as follows: As all separable, infinite-dimensional Frechet spaces are homeomorphic, it suffices to consider $s$, which is homeomorphic to $Q^0$, the countably infinite Cartesian product of the open interval $(-1, 1)$ with itself and a naturally imbedded subset of the Hilbert cube. Let $T$ be any closed, locally compact subset of $Q^0$. There exists a homeomorphism $h$ of $Q$ onto itself carrying $Q^0-T$ onto $Q^0$ [2, Corollary 5.4].

For any (relatively) open, nonnull, homotopically trivial set $U$ in $Q^0$, the set $U'=Q-\mathrm{cl}(Q^0-U)$ is open, nonnull, and homotopically trivial in $Q$. Hence, $h(U')\cdot Q^0=h(U-T)$ is nonnull, (relatively) open in $Q^0$, and homotopically trivial [3, Lemma 8.3], and thus $U-T$ is, also.

In the case of a separable, locally compact group, one may drop the hypothesis of Theorem 2 that the subset on which it acts be locally compact, as Theorem 3 below shows.

THEOREM 3. *Any transformation group action of a separable, locally compact group on a closed set with property $Z$ in a separable, infinite-dimensional Frechet space can be extended to an action of $G$ on the entire space.*

PROOF. Let $(G, X, v)$ be the transformation group under consideration. Without loss of generality, one may assume that the action is effective, that is, that if $g$ is in $G$ and is not the identity, then there is some $x$ in $X$ such that $v(g, x)\neq x$. Effectiveness implies, in this case, that $G$ is metrizable, for it must be Hausdorff and satisfy the first axiom of countability at each point.

As $G$ is separable, locally compact, and metrizable, the space $E$ of all continuous functions from $G$ into $l_2$ under the topology of uniform convergence on compacta is itself a separable, infinite-dimensional Frechet space. Furthermore, the homomorphism of $G$ into the group of linear homeomorphisms of $E$ onto itself defined in the proof of Theorem 1 in this case defines an action of $G$ on $E$ as a transformation group.

Let $f_1$ be an imbedding of $X$ in $l_2$ as a closed set, and let $f_2$ be the imbedding of $X$ in $E$ induced from $f_1$ as in the proof of Theorem 1. Now, as before, the action of $G$ on $E$ extends the action of $G$ on $f_2(X)$ induced from $v$.

Let $Y'=E\times Y$, where $Y$ is the separable, infinite-dimensional

Frechet space containing $X$ which is in question. Now let the action of $G$ on $E$ generate an action of $G$ on $Y'$, trivial in the $Y$ coordinate, as was done in the proof of Theorem 2. If $f_3$: $X \to Y'$ is defined by $f_3(x) = (f_2(x), 0)$, then $f_3$ is an imbedding and $f_3(X)$ has property $Z$ in $Y'$, as it lies in a subspace of infinite co-dimension. As the defined action of $G$ on $Y'$ extends the induced action on $f_3(X)$ (induced from $v$ by $f_3$) and as $Y'$ is a separable, infinite-dimensional Frechet space and is thus homeomorphic to $Y$, there remains only to apply Theorem A in order to complete the proof.

REMARK. Raymond Y.-T. Wong has proven in [12] (but not specifically stated) that if $X$ is a subset of the Hilbert cube $Q$ which is homeomorphic to $Q$ and is bi-collared (that is, there exists an open set $U$ in $Q$ containing $X$ which is homeomorphic to $X \times (-1, 1)$ by a homeomorphism carrying each $x$ in $X$ to $(x, 0)$), then any action of a transformation group on $X$ can be extended to an action on $Q$. (Wong proved that there exists a homeomorphism of $Q$ onto itself carrying $X$ onto the set of all points of $Q$ whose first coordinate is 0.) Wong has also found a Cantor set in $Q$ and a homeomorphism of it onto itself which cannot be extended to a homeomorphism of $Q$ onto itself [11]. These two results are the only ones of which the author is aware concerning the extension of transformation group actions on subsets of $Q$ which do not have property $Z$.

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