AN AXIOMATIC APPROACH TO THE BOUNDARY THEORIES OF WIENER AND ROYDEN

BY PETER A. LOEB AND BERTRAM WALSH[1]

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In this note we announce results, obtained in the framework of Brelot's axiomatic potential theory, which are applicable to the Wiener and Royden boundary theories for Riemann surfaces.[2] Recall that in Brelot's theory, we consider a sheaf $\mathcal{H}$ of real-valued functions with open domains contained in a locally compact, noncompact, connected and locally connected Hausdorff space W, with the functions satisfying certain axioms. Specifically, by a harmonic class of functions on W we mean a class $\mathcal{H}$ of real-valued continuous functions with open domains. For each open $\Omega \subseteq W$, $\mathcal{H}_\Omega$ denotes the set of functions in $\mathcal{H}$ with domains equal to Ω; it is assumed that $\mathcal{H}_\Omega$ is a real vector space. The three axioms of Brelot which $\mathcal{H}$ is assumed to satisfy are (1) a function is in $\mathcal{H}$ if and only if it is locally in $\mathcal{H}$; (2) there is a base for the topology of W which consists of regions regular for $\mathcal{H}$, i.e. connected open sets ω such that any continuous function f on $\partial\omega$ has a unique continuous extension in $\mathcal{H}_\omega$ which is nonnegative if f is nonnegative; (3) the upper envelope of any increasing sequence of functions in $\mathcal{H}_\Omega$ where Ω is a region (i.e. open and connected) is either $+\infty$ or an element of $\mathcal{H}_\Omega$.

Let $\mathcal{H}^-$ and $\mathcal{H}_-$ denote the classes of functions which are superharmonic and subharmonic with respect to $\mathcal{H}$; let $\mathcal{H}^{-b}$ denote the subclass of $\mathcal{H}^-$ consisting of functions bounded below. We assume as another axiom: (4) $\mathbf{1} \in \mathcal{H}_W^-$.

1. Let $\overline{W}$ be a Hausdorff space in which W is imbedded as a dense (and therefore open) subspace, and henceforth let us agree that $\overline{\Omega}$ will mean the closure of Ω in $\overline{W}$ and $\partial\Omega = \overline{\Omega} - \Omega$. If Ω is an open subset of W, we shall say that $\partial\Omega$ *is associated with* $\mathcal{H}_\Omega^{-b}$ if every $v \in \mathcal{H}_\Omega^{-b}$ whose limit inferior is nonnegative at every point of $\partial\Omega$ is necessarily nonnegative on Ω. Throughout this note, we shall denote $\lim_{x\in\Omega, x\to x_0} f(x)$ by $\lim_\Omega f(x_0)$; similar notation is used for lim inf and lim sup.

THEOREM 1.1. *If Ω is an open subset of W and ∂W is associated with $\mathcal{H}_W^{-b}$, then $\partial\Omega$ is associated with $\mathcal{H}_\Omega^{-b}$.*

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[2] These results will appear with proofs as part of a forthcoming article in the Annales de l'Institut Fourier.

Assume that ∂W is associated with $\mathcal{H}_W^{-b}$; then given a bounded real-valued function f on $\partial\Omega$ (where Ω is an open subset of W) one can define $H^-(f, \Omega) \in \mathcal{H}$ to be the lower envelope of the set $\{v \in \mathcal{H}_\Omega^{-b}$: $\lim\inf_\Omega v(x) \geqq f(x)$ for all $x \in \partial\Omega\}$ and dually define $H_-(f, \Omega) = -H^-(-f, \Omega)$. $H^-(f, \Omega)$ and $H_-(f, \Omega)$ are respectively the *upper-* and *lower-*$\mathcal{H}$-extensions of f in Ω. If they are equal, we say that f is *resolutive* on $\partial\Omega$. A point $x_0 \in \partial\Omega$ for which $\lim\sup H^-(f, \Omega)(x_0) \leqq \lim\sup f(x_0)$ for every bounded function f on $\partial\Omega$ is said to be *regular* (with respect to $\mathcal{H}$). Given $x_0 \in \partial\Omega$, a positive function $b \in \mathcal{H}^-$ defined in the intersection of Ω with an open neighborhood of x_0 and for which $\lim_\Omega b(x_0) = 0$ is called an *$\mathcal{H}$-barrier* (or simply a *barrier*) *for Ω at x_0*. We say that there is a *system of barriers* for Ω (or, for emphasis, $\bar{\Omega}$) at x_0 if there is a base $\mathcal{O}$ for the neighborhood system of x_0 such that on the intersection of Ω with $\omega \in \mathcal{O}$ there is defined a barrier b for Ω at x_0 with

$$\inf \{\lim\inf_\Omega b(x_1) : x_1 \in \partial(\omega \cap \Omega) - (\omega \cap \partial\Omega)\} > 0.$$

Such a barrier is said to *belong to Ω and ω*. An $\mathcal{H}_-$-unit-barrier for Ω at x_0 is a function $b_1 \in \mathcal{H}_-$, defined on the intersection of Ω with a neighborhood of x_0 and such that $\lim_\Omega b_1(x_0) = 1$. With these definitions, we have

THEOREM 1.2. *Let x_0 be a point of $\partial\Omega$. Assume there is a system of barriers and an $\mathcal{H}_-$-unit-barrier for Ω at x_0. Then x_0 is a regular point for Ω.*

2. Let $\mathcal{H}$ be a harmonic class which is hyperbolic on W [**5**, p. 189], and let $\mathcal{BH}_W$ denote the set of all bounded $\mathcal{H}$-harmonic functions on W. Then $\mathcal{BH}_W$ is a Banach lattice with order unit $H(W)$, where $H(W)$ is the greatest $\mathcal{H}$-harmonic minorant of **1**. The lattice operation $\vee_{\mathcal{H}}$ is given by defining $f \vee_{\mathcal{H}} g$ to be the least $\mathcal{H}$-harmonic majorant of the pointwise supremum $f \vee g$, and $\wedge_{\mathcal{H}}$ is similarly defined.

We next consider ideal boundary theory for an arbitrary Banach sublattice $\mathfrak{H}$ of $\mathcal{BH}_W$ when $H(W) \in \mathfrak{H}$. Some examples of such sublattices are:

(1) $\mathcal{BH}_W$ itself.

(2) The uniform closure of the space $\mathcal{BDH}_W$, where $\mathcal{BDH}_W$ is the set of all bounded harmonic functions (in the usual sense) with finite Dirichlet integral on an open Riemann surface W.

(3) The uniform closure of the space of all bounded C^2-functions f on an open Riemann surface W such that:

(a) $\Delta f = Pf$ where P is a nonnegative density on W with $\iint_W P < \infty$, and

(b) $D(f,f)+\int\int_W Pf^2<\infty$ where $D(f,f)$ is the Dirichlet integral of f.

Let a Banach sublattice $\mathfrak{H}$ of $\mathcal{B}\mathcal{H}_W$ containing the order unit, $H(W)$, be given. Now form the Q-compactification [2, pp. 96–97] and [6] $W^*_{\mathfrak{H}}$ of W with $Q=\mathfrak{H}$; this is a compact Hausdorff space containing W as a dense subspace, determined up to homeomorphism by the properties that each $f\in\mathfrak{H}$ has a continuous extension to $W^*_{\mathfrak{H}}$ and that the family of all these extensions separates the points of $\Delta_{\mathfrak{H}}=W^*_{\mathfrak{H}}-W$. Define

$$\Gamma_{\mathfrak{H}} = \{t\in\Delta_{\mathfrak{H}}:H(W)(t) = 1\}\cap\bigcap_{f,g\in\mathfrak{H}}\{t\in\Delta_{\mathfrak{H}}:(f\wedge_{\mathcal{H}}g)(t) = (f\wedge g)(t)\}$$

and let $\overline{W}_{\mathfrak{H}}=W\cup\Gamma_{\mathfrak{H}}$. Then

THEOREM 2.1. *$\Gamma_{\mathfrak{H}}$ is associated with $\mathcal{H}^{-b}_W$, whence $\Gamma_{\mathfrak{H}}$ is nonempty.*

THEOREM 2.2. *If $M\subseteq\Delta_{\mathfrak{H}}$ is a closed set which is associated with $\mathcal{H}^{-b}_W$, then the restriction map $f\to f|M$ of $\mathfrak{H}$ into $\mathcal{C}_R(M)$ is an isometry (not necessarily onto) preserving positivity in both directions.*

Now by the lattice form of the Stone-Weierstrass theorem we have

THEOREM 2.3. *The restriction mapping $f\to f|\Gamma_{\mathfrak{H}}$ of $\mathfrak{H}$ into $\mathcal{C}_R(\Gamma_{\mathfrak{H}})$ is a surjective isometry sending the order unit of $\mathfrak{H}$ to the order unit* **1** *of $\mathcal{C}_R(\Gamma_{\mathfrak{H}})$ and preserving the lattice operations.*

THEOREM 2.4. *$\Gamma_{\mathfrak{H}}$ is the intersection of all sets $\Gamma_p=\{t\in\Delta_{\mathfrak{H}}:$ $\liminf p(t)=0\}$ as p ranges through the $\mathcal{H}$-potentials on W. No proper closed subset of $\Gamma_{\mathfrak{H}}$ is associated with $\mathcal{H}^{-b}_W$.*

THEOREM 2.5. *Except perhaps when $\mathfrak{H}$ consists only of constant functions, there is an $\mathcal{H}_-$-unit barrier and a system of barriers for $W^*_{\mathfrak{H}}$ at each point of $\Gamma_{\mathfrak{H}}$, whence each $x\in\Gamma_{\mathfrak{H}}$ is regular with respect to any open set $\Omega\subset W$ for which $x\in\partial\Omega\cap\Gamma_{\mathfrak{H}}$. (Here $\partial\Omega$ is taken in $W^*_{\mathfrak{H}}$.)*

THEOREM 2.6. *Let $\tilde{\mathfrak{H}}$ denote those bounded functions in $\mathcal{H}^-_W$ for which the greatest $\mathcal{H}$-harmonic minorant is in $\mathfrak{H}$. For any $v\in\tilde{\mathfrak{H}}$, let $I(v)$ be the function on $\Gamma_{\mathfrak{H}}$ defined by $I(v)(t)=\liminf_W v(t)$ for each $t\in\Gamma_{\mathfrak{H}}$. Then $I(v)$ is continuous on $\Gamma_{\mathfrak{H}}$ for each $v\in\tilde{\mathfrak{H}}$, and the mapping I: $\tilde{\mathfrak{h}}\to\mathcal{C}_R(\Gamma_{\mathfrak{H}})$ is positively homogeneous and additive.*

If W is an open Riemann surface, $\mathcal{H}$ the class of harmonic functions in the usual sense, and $\mathfrak{H}=\mathcal{B}\mathcal{H}_W$, then $\Gamma_{\mathfrak{H}}$ is homeomorphic to the harmonic part of the Wiener boundary even though $\Delta_{\mathfrak{H}}$ is "smaller" than the Wiener boundary. If $\mathfrak{H}$ is the uniform closure of $\mathcal{B}\mathcal{D}\mathcal{H}_W$, the bounded harmonic functions with finite Dirichlet integrals, then $\Gamma_{\mathfrak{H}}$

is the harmonic part of the Royden boundary and $\mathfrak{BDH}_W$ is isometrically isomorphic to a dense subset of $\mathcal{C}_R(\Gamma_{\mathfrak{H}})$.

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University of California, Los Angeles