BOUNDARY VALUE PROBLEMS FOR DELAY-DIFFERENTIAL EQUATIONS

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1. **Introduction.** In this note we shall give some sufficient conditions for the existence of solutions of a certain type of boundary value problem (BVP) for delay-differential equations (d.d.e.'s). The conditions given are of two kinds, in Theorem 1 a relationship between the boundary conditions and the size of the interval under consideration implies the existence of solutions; in Theorem 4 the existence of solutions of delay-differential inequalities implies the existence of solutions. A discussion concerning the formulation of BVP's of the type considered here may be found in [1], [2], and [3]; these sources in turn reveal much of the literature concerning such problems.

2. **The problem.** Let f be a real-valued continuous function defined on $R^{n+m+2}\times I$, where I is the compact interval $[a, b]$. Let $h_1(t), \cdots, h_n(t), g_1(t), \cdots, g_m(t)$ be nonnegative continuous functions with domain I. Assume that $t-g_i(t)$ assumes the value a at most a finite number of times as t ranges over I and $i=1, \cdots, m$. Define the real number c by

$$c = \min \left\{ \min_{1\leqq i\leqq n} \inf_{t\in I} (t - h_i(t)), \min_{1\leqq j\leqq m} \inf_{t\in I} (t - g_j(t))\right\}$$

and let $J=[c, a]$. Let $\phi(t)\in C^1(J)$ and let B be any real number; we then seek a function $x(t)\in C(J\cup I)\cap C^1(J)\cap C^1(I)$ having a piecewise continuous second derivative such that

$$(1)\quad x(t) = \phi(t), \qquad x'(t) = \phi'(t), \qquad t\in J, \qquad x(\bar{b}) = B, \qquad \bar{b} \leqq b.$$

and

$$(2)\quad x''(t) = f(x(t), x(t-h_1(t)), \cdots, x(t-h_n(t)), x'(t), x'(t-g_1(t)), \cdots, x'(t-g_m(t), t)$$

for $a \leqq t \leqq \bar{b}$.

In general we must expect that a solution of problem (1)—(2) will have a discontinuous derivative at $t=a$, and therefore the second derivative will in general only be piecewise continuous if the right side of (2) depends on delays in x'.

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3. Existence results. Consider now the BVP (1)–(2).

THEOREM 1. *Let $M>0$, $N>0$ be given and let*

$$Q = \sup \{ |f(x_1, \cdots, x_{n+m+2}, t)| : |x_i| \leqq 2M, i = 1, \cdots, n+1; \\ |x_j| \leqq 2N, j = n+2, \cdots, n+m+2; a \leqq t \leqq b \}.$$

Then if $\bar{b}$, $a<\bar{b}\leqq b$, is chosen so that

$$\bar{b} - a \leqq \min \{ (8M/Q)^{1/2}, 2N/Q \},$$

BVP (1)–(2) has a solution for any $\phi \in C^1(J)$ with $|\phi(t)| \leqq M$, $|\phi'(t)| \leqq N$ and any real number B, $|B| \leqq M$ and

$$|(\phi(a) - B)/(\bar{b} - a)| \leqq N.$$

The proof of Theorem 1 may be obtained by means of the Schauder-Tychonoff Fixed Point Theorem in the following way. We define a mapping T from the Banach space

$$(B, \|\cdot\|) = (C[c, \bar{b}] \cap C^1[c, a] \cap C^1[a, \bar{b}], \|\cdot\|),$$

where

$$\|x\| = \sup_{c \leqq t \leqq \bar{b}} |x(t)| + \max \{ \sup_{c \leqq t \leqq a} |x'(t)|, \sup_{a \leqq t \leqq \bar{b}} |x'(t)| \},$$

into B by

$$Tx(t) = \int_a^{\bar{b}} \bar{G}(t; s) f(x(s), \cdots, x'(s), \cdots, s) ds + l(t)$$

where

$$\begin{aligned} \bar{G}(t; s) &= G(t; s), \quad a \leqq t \leqq \bar{b}, \\ &= 0, \quad c \leqq t \leqq a, \end{aligned} \qquad a \leqq s \leqq \bar{b},$$

$G(t; s)$ is the Green's function with respect to the BVP

$$x'' = 0, \qquad x(a) = 0 = x(\bar{b})$$

and $l(t)$ is the function

$$\begin{aligned} l(t) &= \phi(t), \qquad c \leqq t \leqq a, \\ &= \frac{B - \phi(a)}{\bar{b} - a}(t - a) + \phi(a), \qquad a \leqq t \leqq \bar{b}. \end{aligned}$$

One may then show that T has a fixed point. Fixed points of T, however, are solutions of BVP (1)–(2).

The following corollary is important in the proof of the results to follow.

COROLLARY 2. *Assume there exists a constant Q such that $|f| \leqq Q$ on $R^{n+m+2} \times I$. Then any BVP* (1)–(2) *has a solution.*

DEFINITION. A function $\alpha(t) \in C(J \cup I) \cap C^1(J) \cap C^1(I)$ having a piecewise continuous second derivative is called a lower solution with respect to BVP (1)–(2) provided

(*i*) $\alpha(t) \leqq \phi(t), \quad t \in J, \quad \alpha(b) \leqq B,$

(*ii*) $\alpha''(t) \geqq f(\alpha(t), \alpha(t - h_1(t)), \cdots, \alpha'(t), \alpha'(t - g_1(t)), \cdots, t)$ for $a \leqq t \leqq b$.

An upper solution β of (1)–(2) is defined by reversing the inequalities in (i) and (ii).

Consider now the d.d.e.

$$x''(t) = f(x(t), x(t - h_1(t)), \cdots, x(t - h_n(t)), x'(t), t). \tag{3}$$

LEMMA 3. *Let there exist a constant Q such that $|f| \leqq Q$. Let α and β be lower and upper solutions of BVP* (1)–(3) *with $\alpha(t) \leqq \beta(t)$ for $t \in I$. Furthermore, assume that f is nonincreasing in the second through $(n+1)$st argument. Then there exists a solution $x(t)$ of BVP* (1)–(3) *such that $\alpha(t) \leqq x(t) \leqq \beta(t)$ for $t \in I$.*

Making use of Lemma 3 we may now obtain results for d.d.e.'s of the form (3) and

$$x''(t) = f(x(t), x(t - h_1(t)), \cdots, x(t - h_n(t)), t). \tag{4}$$

THEOREM 4. *Let f be nonincreasing in the second through $(n+1)$st argument. Then BVP* (1)–(4) *has a solution if and only if there exist lower and upper solutions α and β of* (1)–(4) *with $\alpha(t) \leqq \beta(t)$ on I.*

This theorem is very useful in many instances where lower and upper solutions may easily be found. Consider e.g. the following BVP:

$$x(t) = \phi(t), \quad c \leqq t \leqq a, \quad x(b) = B, \tag{5}$$

$$x''(t) = x(t) - x(t - h(t)), \quad a \leqq t \leqq b. \tag{6}$$

Then it is clear that

$$\beta = \max \left\{ \sup_{c \leqq t \leqq a} \phi(t), B \right\} \quad \text{and} \quad \alpha = \min \left\{ \inf_{c \leqq t \leqq a} \phi(t), B \right\}$$

are upper and lower solutions of (5)–(6). Hence there exists a solution $x(t)$ of (5)–(6) such that $\alpha \leqq x(t) \leqq \beta$.

Results similar to Theorem 4 for BVP (1)–(3) may be obtained provided some condition is imposed on f which guarantees a bound on the derivative of a solution in terms of a bound on the solution. For example if f satisfies a growth condition

$$|f| \leqq C_1 + C_2 |x'|^2$$

where C_1 and C_2 are nonnegative functions of the remaining arguments, then the existence of lower and upper solutions α and β, $\alpha(t) \leqq \beta(t)$, implies the existence of a solution of BVP (1)–(3).

Proofs of the above results and other existence theorems concerning such BVP's and periodic solutions of d.d.e.'s will appear elsewhere.

References

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