PERTURBING ASYMPTOTICALLY STABLE DIFFERENTIAL EQUATIONS

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Our purpose here is to announce new theorems on the eventual uniform-asymptotic stability (hereafter called EvUAS) of the origin 0 for the ordinary differential equation

$$\text{(P)} \qquad x' = f(t, x) + g(t, x), \qquad (x' = dx/dt)$$

given that 0 is EvUAS for the equation

$$\text{(E)} \qquad x' = f(t, x),$$

and that f and g satisfy certain conditions. We always assume that f and g are at least continuous from $[0, \infty) \times R^d$ to R^d, but we never assume that the solutions of (P) are unique or that the zero function is a solution of (P). In fact EvUAS is a natural generalization of uniform asymptotic stability in which it is not assumed that the zero function is a solution.

Our main result is (definitions follow)

THEOREM A. *Let* 0 *be EvUAS for* (E). *Then* 0 *is EvUAS for* (P) *if*

(i) *f is Lipschitz and g is diminishing, or*

(ii) *f is periodic and g is diminishing, or*

(iii) *f is inner product and g is absolutely diminishing, or*

(iv) *f is linear and $g = g_1 + g_2$, where g_1 is absolutely diminishing and* $g_2 = o(|x|)$.

Let $x(t; t_0, x_0)$ denote a solution of (E) through (t_0, x_0). We say that 0 is EvUAS for (E) if

$$\lim_{t_0 \to \infty;\, |x_0| \to 0} \left[\sup_{t \geqq t_0} |x(t; t_0, x_0)|\right] = 0$$

and if, for some $\delta_0 > 0$ and some $\alpha_0 \geqq 0$,

$$\lim_{t \to \infty} \left[\sup_{t_0 \geqq \alpha_0;\, |x_0| < \delta_0} |x(t + t_0; t_0, x_0)|\right] = 0.$$

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We say that f is *Lipschitz* if, for some $r>0$ and $L>0$,

$$|f(t, x) - f(t, y)| \leqq L|x - y| \quad \text{for } t \geqq 0, \quad |x| \leqq r, \quad \text{and} \quad |y| \leqq r;$$

inner product if, for some $r>0$ and $L>0$,

$$\langle x-y, f(t, x)-f(t, y)\rangle \leqq L\|x-y\|^2 \quad \text{for } t\geqq 0, \quad |x| \leqq r, \quad \text{and} \quad |y| \leqq r;$$

linear if

$$f(t, x) = A(t)x \quad \text{for } t \geqq 0 \quad \text{and} \quad x \in R^d;$$

and *periodic* if, for some $\omega>0$,

$$f(t + \omega, x) = f(t, x) \quad \text{for } t \geqq 0 \quad \text{and} \quad x \in R^d.$$

Note that if $A(t)$ is bounded on $[0, \infty)$, then $f(t, x)=A(t)x$ is both linear and Lipschitz; that f is periodic if it is independent of t; and that a Lipschitz function is also inner product.

We say that g is *absolutely diminishing* if, for some $r>0$ and every m satisfying $0<m<r$, there exists a function h_m such that, for all $t\geqq 0$ and $m\leqq |x| \leqq r$,

$$|g(t, x)| \leqq h_m(t) \quad \text{and} \quad \int_t^{t+1} h_m(s)ds \to 0 \quad \text{as } t \to \infty.$$

We say that g is *diminishing* if: (i) g is absolutely diminishing; or (ii) g is continuous in x uniformly with respect to $t\in[0, \infty)$ and, for some $r>0$ and each fixed x satisfying $0<|x|<r$,

$$\sup_{0\leqq u\leqq 1}\left|\int_t^{t+u} g(s, x)ds\right| \to 0 \quad \text{as } t \to \infty;$$

or (iii) g is a finite sum of functions of types (i) and (ii).

Theorem A generalizes the following result, obtained in stages by Malkin [3, p. 104], Vrkoc [7], Wexler [8], Yoshizawa [9], Krasovskii [4, p. 102], LaSalle and Rath [5], and Strauss and Yorke [6]:

THEOREM. *If* 0 *is UAS for* (E), *if f is Lipschitz, and if g is absolutely diminishing, then* 0 *is EvUAS for* (P).

Theorem A also generalizes the following result, obtained in stages by Poincaré, Liapunov, Perron, Coddington and Levinson [2, p. 327], Brauer [1], and Strauss and Yorke [6]:

THEOREM. *Let A be a constant matrix. If* 0 *is UAS for $x'=Ax$ and if $g=g_1+g_2$, where g_1 is absolutely diminishing* (*but with* $m=0$) *and* $g_2=o(|x|)$, *then* 0 *is "eventually asymptotically stable" for* (P).

There do not seem to be any results in the literature for f merely periodic or inner product,

We now briefly discuss diminishing functions. If $|g(t, x)| \leqq h(t)$ for all $t \geqq 0$ and $|x| \leqq r$, then g is absolutely diminishing whenever, in particular, $h(t) \to 0$ as $t \to \infty$ or

$$\int_0^\infty |h(t)|^p dt < \infty \quad \text{for some } p \geqq 1.$$

The scalar function $g(t, x) = t(t^2x^2+1)^{-1}$ is absolutely diminishing because we may choose

$$h_m(t) = t(t^2m^2 + 1)^{-1} \to 0 \quad \text{as } t \to \infty;$$

however, $g(t, 0) = t$. Thus an absolutely diminishing function need not be small at $x=0$. The function

$$h(t) = (t \sin t^3, t \cos t^3, 0, \cdots, 0)$$

is diminishing but not absolutely diminishing, since $\|h(t)\| = t$. Furthermore, $t^{-1}h(t)$ is bounded and diminishing, but not absolutely diminishing. If $k(x)$ is continuous from R^d to R^d (we need not have $k(0)=0$), then the function $k(x) \sin t^3$ is diminishing but not absolutely diminishing. Examples show that if uniform continuity is dropped from the definition of a diminishing function, then part (i) of Theorem A may fail.

We now summarize some of our other results.

THEOREM B. *Let* 0 *be EvUAS for* (E). *Let f be Lipschitz or periodic. Then* 0 *is EvUAS for*

$$x' = f(t, x) + h(t) \tag{1}$$

if and only if h is diminishing. In fact if h is not diminishing, then no solution of (1) *can approach zero as* $t \to \infty$.

Both implications of Theorem B are false for inner product f and for linear f. Furthermore, there exist diminishing functions g and there exist functions f which are both inner product and linear such that 0 is EvUAS for (E) but not for (P).

To understand better the relationship between properties of f in (E) and the conditions for admissible perturbations g, we use the concept of perturbation classes. Define

$$\mathfrak{F}_C = \{f(t, x): f \text{ is continuous from } [0, \infty) \times R^d \text{ to } R^d\}.$$

Let $\mathfrak{F}$ be a subclass of $\mathfrak{F}_C$. Define the *perturbation classes*

$$\mathcal{G}(\mathfrak{F}) = \{g \in \mathfrak{F}_C: \forall f \in \mathfrak{F},\ 0 \text{ is EvUAS for (E)} \Rightarrow 0 \text{ is EvUAS for (P)}\},$$
$$\mathcal{H}(\mathfrak{F}) = \{h \in \mathcal{G}(\mathfrak{F}): h \text{ is independent of } x\}.$$

Then if $\mathfrak{F}_{\text{Lip}}$ denotes the class of Lipschitz functions, $\mathfrak{F}_{\text{Inn}}$ the class of inner product functions, $\mathfrak{F}_{\text{Lin}}$ the class of linear functions, and $\mathfrak{F}_{\text{Per}}$ the class of periodic functions, we may restate Theorem A as

$$\mathcal{G}(\mathfrak{F}_{\text{Lip}}) \supset \{g(t, x): g \text{ is diminishing}\},$$
$$\mathcal{G}(\mathfrak{F}_{\text{Per}}) \supset \{g(t, x): g \text{ is diminishing}\},$$
$$\mathcal{G}(\mathfrak{F}_{\text{Inn}}) \supset \{g(t, x): g \text{ is absolutely diminishing}\},$$
$$\mathcal{G}(\mathfrak{F}_{\text{Lin}}) \supset \{g_1 + g_2: g_1 \text{ is absolutely diminishing and } g_2 = o(|x|)\}.$$

Theorem B implies

$$\mathcal{H}(\mathfrak{F}_{\text{Lip}}) = \mathcal{H}(\mathfrak{F}_{\text{Per}}) = \{h(t): h \text{ is diminishing}\}.$$

The remarks following Theorem B imply $\mathcal{H}(\mathfrak{F}_{\text{Lip}}) \neq \mathcal{H}(\mathfrak{F}_{\text{Lin}})$ and $\mathcal{H}(\mathfrak{F}_{\text{Lip}}) \neq \mathcal{H}(\mathfrak{F}_{\text{Inn}})$.

The conditions we impose on g are that g be "small as $t \to \infty$." We can use conditions of the type "g is small as $|x| \to 0$" and still perturb every equation (E) for f linear, but not for f Lipschitz, inner product, or periodic, as Theorem A and the next result show.

THEOREM C. *Let $d \geqq 2$. Then we have the following:*

for Lipschitz $g(x)$, $g \in \mathcal{G}(\mathfrak{F}_{\text{Lip}}) \Leftrightarrow g(x) \equiv 0$ *near* $x = 0$;

for Lipschitz $g(x)$, $g \in \mathcal{G}(\mathfrak{F}_{\text{Inn}}) \Leftrightarrow g(x) \equiv 0$ *near* $x = 0$;

for continuous $g(x)$, $g \in \mathcal{G}(\mathfrak{F}_{\text{Per}}) \Leftrightarrow g(x) \equiv 0$ *near* $x = 0$;

for a constant matrix A, $Ax \in \mathcal{G}(\mathfrak{F}_{\text{Lin}}) \Leftrightarrow Ax = \alpha x$ *for some* $\alpha \leqq 0$.

Finally, we show that restrictions on f (such as Lipschitz, etc.) are needed in order to prove a result like Theorem A. Let $\mathfrak{F}_{CU}$ be the class of functions which are locally Lipschitz and uniformly continuous on $[0, \infty) \times R^d$.

THEOREM D. *For some* $f \in \mathfrak{F}_{CU}$, 0 *is EvUAS for* (E) *but not for*

$$x' = f(t, x) + e^{-t}(1, \cdots, 1).$$

Also, for some $f \in \mathfrak{F}_{CU}$, 0 *is EvUAS for* (E) *but not for*

$$x' = f(t, x) + xe^{-t}.$$

In particular, then, $e^{-t}(1, \cdots, 1) \notin \mathcal{G}(\mathfrak{F}_{CU})$ *and* $xe^{-t} \notin \mathcal{G}(\mathfrak{F}_{CU})$.

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