REGULAR NEIGHBORHOODS ARE NOT TOPOLOGICALLY INVARIANT

BY MITSUYOSHI KATO

Communicated by W. Browder, May 1, 1968

In [5], Siebenmann and Sondow have shown that there exist topologically equivalent PL$(n+3, n+1)$-sphere pairs for $n \geqq 2$ that are combinatorially distinct. In this note, combining their analysis of strong h-cobordisms of certain higher dimensional knots and isoneighboring theorem due to Noguchi [3] and [4], we show the following:

THEOREM. *Assume $n = \text{even} \geqq 2$. Then there exist infinitely many combinatorially distinct* PL$(n+3, n+1)$*-manifold pairs (V_k, K_k), $k = 1, 2, \cdots$, that are not abstract regular neighborhoods but topologically equivalent to an abstract regular neighborhood (V_0, K_0).*

REMARK. Each submanifold K_k is a PL$(n+1)$-sphere which is 1-flat in V_k with only one singularity. (For 1-flat embeddings and singularities, see [3].)

An implication of the Theorem is that regular neighborhoods are not topologically invariant. More explicitly we may say:

COROLLARY. *The collapsing is not topologically invariant.*

We note here that (V_k, K_k) and (V_0, K_0) have the vanishing Whitehead torsion, since K_0 is simply connected. However, in the subsequent paper [2], we shall show that the topological invariance of Whitehead torsions is equivalent to that of regular neighborhoods of polyhedra in the sufficiently high-dimensional euclidean space.

1. The construction. In the following, we shall use the notations in [5]. However, we shall be concerned mainly with the combinatorial (or PL) objects. By a PL *n-knot* we shall mean a PL$(n+2, n)$-sphere pair (S^{n+2}, L^n) such that L^n has a collar neighborhood $(L^n \times D^2)$ in S^{n+2} [1] and [4].

LEMMA 1. *Assume $n = \text{even} \geqq 2$. Then there exist infinitely many invertible strong h-cobordisms of* PL *n-knots*

$c_k = ((W_k, M_k); (S_0, L_0), (S_k, L_k))$, $k = 1, 2, \cdots$, such that

(1) (S_k, L_k) and (S_0, L_0) are combinatorially equivalent,

(2) $\pi_1(S_0 - L_0) \cong J \times G$, where J and G are the infinite cyclic group and the binary icosahedral group, respectively, and

(3) $\tau(c_k)=2k\tau$ *for* $k=1, 2, \cdots$, *where* τ *is an element of* $\mathrm{Wh}(J\times G)$ *of infinite order such that when* $k\neq j$, *there exists no automorphism* θ *of* $\pi_1(S_0-L_0)$ *making* $\theta_*\tau(c_k)=\tau(c_j)$.

The proof of Lemma 1 is essentially given in [5]. In particular, it is to be noted that the argument in Construction 2.5 of [5] is valid for $n=\text{even}\geqq 2$.

In the subsequent, we shall employ the notations in Lemma 1 and assume $n=\text{even}\geqq 2$. First, we form a PL$(n+3, n+1)$-manifold pair $(V_0, K_0)=a*(S_0, L_0)\cup_{f_0}(D^{n+1}\times D^2, D^{n+1}\times(0))$ from the cone ball pair $a*(S_0, L_0)(=(a*S_0, a*L_0))$ by attaching the standard ball pair $(D^{n+1}\times D^2, D^{n+1}\times(0))$ by a PL homeomorphism

$$f_0\colon (bD^{n+1}\times D^2, bD^{n+1}\times(0))\to((L_0\times D^2), L_0)\subset(S_0, L_0),$$

where bD^{n+1} stands for the boundary of the $(n+1)$-ball D^{n+1}. Then (V_0, K_0) is clearly an abstract regular neighborhood and K_0 is 1-flat in V_0 with only one singularity (S_0, L_0) at a.

In the same way, we have a PL$(n+3, n+1)$-manifold pair $(V_k, K_k)=a*(S_0, L_0)\cup(W_k, M_k)\cup_{f_k}(D^{n+1}\times D^2, D^{n+1}\times(0))$ for each $k\geqq 1$, where $f_k\colon(bD^{n+1}\times D^2, bD^{n+1}\times(0))\to((L_k\times D^2), L_k)\subset(S_k, L_k)$ is a PL homeomorphism.

2. Distinguishing the pairs combinatorially. Let U_k be a regular neighborhood of K_k in V_k such that $U_k\subset\mathrm{Int}\ V_k$ for each $k\geqq 0$. Since K_k is a 1-flat $(n+1)$-sphere in U_k with only one singularity (S_0, L_0) at a for each $k\geqq 0$, it follows from the Theorem in [4] that (U_k, K_k) is combinatorially equivalent to (V_0, K_0). Thus we proved:

ASSERTION 1. *For each* $k\geqq 0$, *the abstract regular neighborhoods* (U_k, K_k) *and* (V_0, K_0) *are combinatorially equivalent.*

Putting $N_k=V_k-\mathrm{Int}\ U_k$, we examine a PL manifold triad $(N_k; bU_k, bV_k)$. From Assertion 1, we may identify (U_k, K_k) with (V_0, K_0). Note that $bV_0=E_0\cup_{f_0'}D^{n+1}\times bD^2$ and hence that $\pi_1(E_0)\cong\pi_1(bV_0)$, where $E_0=S_0-(L_0\times\mathrm{Int}\ D^2)$ and $f_0'=f_0|bD^{n+1}\times bD^2$.

Observe that from the construction of the h-cobordism c_k, V_k is obtained from V_0 by attaching an h-cobordism from bV_0 with the torsion $2k\tau$. Here we identify $\mathrm{Wh}(\pi_1(bV_0))$ with $\mathrm{Wh}(\pi_1(E_0))$ by the isomorphism $\pi_1(E_0)\cong\pi_1(bV_0)$. In particular, by the regular neighborhood annulus theorem $(N_0; bU_0, bV_0)$ is a product cobordism. Therefore, we may conclude the following:

ASSERTION 2. *For each* $k\geqq 0$, *the triad* $(N_k; bU_k, bV_k)$ *is a* PL *h-cobordism with* $\tau(N_k, bU_k)=2k\tau$, *where the torsion is identified by the isomorphism* $\pi_1(E_0)\cong\pi_1(bV_0)\cong\pi_1(bU_0)$.

Now we are ready to prove the following:

PROPOSITION 1. *The manifold pairs* (V_k, K_k), $k=1, 2, \cdots$, *are not abstract regular neighborhoods. If* $k\neq j$, *then* (V_k, K_k) *and* (V_j, K_j) *are combinatorially distinct for* $k\geqq 0$ *and* $j\geqq 0$.

PROOF. First, suppose that (V_k, K_k) is an abstract regular neighborhood for some $k\geqq 1$. Then it follows that by the uniqueness of regular neighborhoods (V_k, K_k) and (U_k, K_k) are combinatorially equivalent. Hence, from Assertion 1, (V_k, K_k) and (V_0, K_0) are combinatorially equivalent. Thus, in order to prove the first statement of Proposition 1, it suffices to show the second statement. To do this, suppose that there exists a PL homeomorphism $g: (V_k, K_k)\rightarrow(V_j, K_j)$ for some $k\geqq 0$ and $j\geqq 0$ such that $k\neq j$. Then by the uniqueness of regular neighborhoods we may assume that $g(U_k)=U_j$ and hence that $g(N_k)=N_j$. Thus the h-cobordisms $(N_k; bU_k, bV_k)$ and $(N_j; bU_j, bV_j)$ are combinatorially equivalent. It follows from the combinatorial invariance theorem of Whitehead torsions that $g'_*(2k\tau)=2j\tau$, where $g'=g|bU_k: bU_k\rightarrow bU_j$. This contradicts Lemma 1, (3), completing the proof.

3. **Finding homeomorphisms.** Form (W_k', M_k') from (W_k, M_k) by attaching a collar $(S_0, L_0)\times[0, 1)$ naturally at the left end (S_0, L_0). Then, from the invertibility of the knot cobordisms we may prove the following by the infinite repetition argument:

LEMMA 2. *For any* $k\geqq 1$, (W_k', M_k') *is* PL *homeomorphic to* $(S_k, L_k)\times[0, 1)$. (For the proof, see Lemma 3.1 in [5].)

From Lemma 2 and the cone extension argument, we have:

COROLLARY TO LEMMA 2. *Any homeomorphism* $h: (S_k, L_k)\rightarrow(S_0, L_0)$ *between the boundaries of* $a*(S_0, L_0)\cup(W_k, M_k)$ *and* $a*(S_0, L_0)$ *is extendable to a homeomorphism* $g: a*(S_0, L_0)\cup(W_k, M_k)\rightarrow a*(S_0, L_0)$.

PROPOSITION 2. *For any* $k\geqq 1$, *there is a homeomorphism* $H: (V_k, K_k)\rightarrow(V_0, K_0)$.

PROOF. Let $h: (S_k, L_k)\rightarrow(S_0, L_0)$ be a PL homeomorphism between the boundaries of $a*(S_0, L_0)\cup(W_k, M_k)$ and $a*(S_0, L_0)$. Then from the uniqueness of regular neighborhoods we may assume that

$$hf_k(bD^{n+1}\times D^2, bD^{n+1}\times(0)) = f_0(bD^{n+1}\times D^2, bD^{n+1}\times(0)).$$

It follows from Theorem C in [1] that

$$f_0^{-1}hf_k: (bD^{n+1}\times D^2, bD^{n+1}\times(0)) \rightarrow (bD^{n+1}\times D^2, bD^{n+1}\times(0))$$

is extendable to a PL homeomorphism

$$g: (D^{n+1} \times D^2, D^{n+1} \times (0)) \rightarrow (D^{n+1} \times D^2, D^{n+1} \times (0)).$$

Combining this fact and the Corollary to Lemma 2, we conclude that the PL homeomorphism $h: (S_k, L_k) \rightarrow (S_0, L_0)$ is extendable to the required homeomorphism $H: (V_k, K_k) \rightarrow (V_0, K_0)$, completing the proof.

Now Propositions 1 and 2 complete the proof of Theorem.

References

1. M. Kato, *A concordance classification of* PL *homeomorphisms of* $S^p \times S^q$, Topology (to appear).

2. ———, *Geometrical operations of Whitehead groups* (to appear).

3. H. Noguchi, *1-flat 3-manifolds in 5-space*. Osaka J. Math. 1 (1964), 117–125.

4. ———, *One-flat submanifolds with codimension two*, Illinois J. Math. (to appear).

5. L. Siebenmann and J. Sondow, *Some homeomorphic sphere pairs that are combinatorially distinct*, Comment. Math. Helv. 41 (1966–67), 261–272.

Tokyo Metropolitan University, Tokyo, Japan