A SPECTRAL SEQUENCE FOR CLASSIFYING LIFTINGS IN FIBER SPACES[1]

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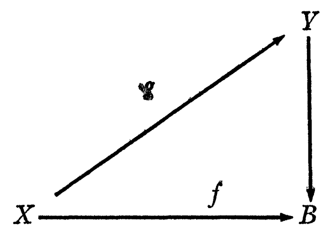
Consider the following diagram of pointed spaces and maps

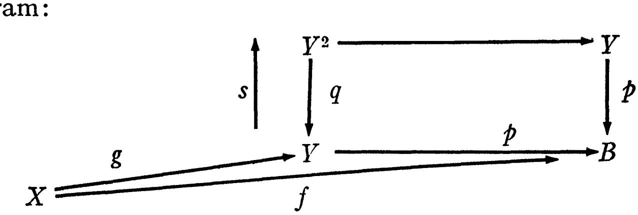


where $pg=f$ and p is a fibration with fiber F. Suppose that X is a CW-complex of dimension $\leqq 2\mathrm{conn}(F)$ and $\mathrm{conn}(F)\geqq 1$ (conn $=$connectivity). Let $[X, Y]_B$ be the set of homotopy classes of pointed maps over f(H: $X\times I\to Y$ is a homotopy over f if $pH_t=f$ for each $t\in I$). Becker proved in [2], [3] that under these hypotheses $[X, Y]_B$ can be given an abelian group structure with $[g]$ as zero element.

The purpose of this note is to describe a spectral sequence of the Adams type which converges to $[X, Y]_B$. The differentials of the spectral sequence are the twisted operations described in [6], [7]. The sequence has the same relation to the method of computing $[X, Y]_B$ used in [6], [7] as the Adams spectral sequence has to the killing-homotopy method of computing ordinary homotopy groups. This note should be read as a sequel to [7].

A different spectral sequence for $[X, Y]_B$ is given by Becker in [3]. A sequence apparently similar to the one to be described here is mentioned in [4] and credited to Becker and Milgram.

1. The spectral sequence. Consider the following commutative diagram:



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where Y^2 is the square of Y, i.e. the pullback of p by p, and s is the canonical cross section. Write (Y^2, Y) for $(Y^2, s(Y))$. Let $A = A_p$ be the mod p Steenrod algebra and use Z_p coefficients for all cohomology. Let $i\colon F \subset Y^2$ and assume that $i^*\colon H^*(Y^2) \to H^*(F)$ is onto. Assume also that $H_j(F; Z)$ is finitely generated for each j. Let $A(Y) = H^*(Y) \odot A$ be the Massey-Peterson algebra [5]. Then $H^*(Y^2, Y)$ and $H^*(X, *)$ are $A(Y)$ modules via $p\colon Y^2 \to Y$ and $g\colon X \to Y$.

THEOREM. *Under the above hypotheses, there is a spectral sequence such that*

(1) $E_2^{s,t} = \mathrm{Ext}_{A(Y)}^{s,t}(H^*(Y^2, Y), H^*(X, *))$
(2) $E_\infty^{s,s} = B^{s,s}/B^{s+1,s+1}$, *where* $[X, Y]_B = B^{0,0} \supset B^{1,1} \supset B^{2,2} \supset \cdots$

and $\bigcap B^{s,s} =$ *all elements of* $[X, Y]_B$ *of finite order prime to* p.

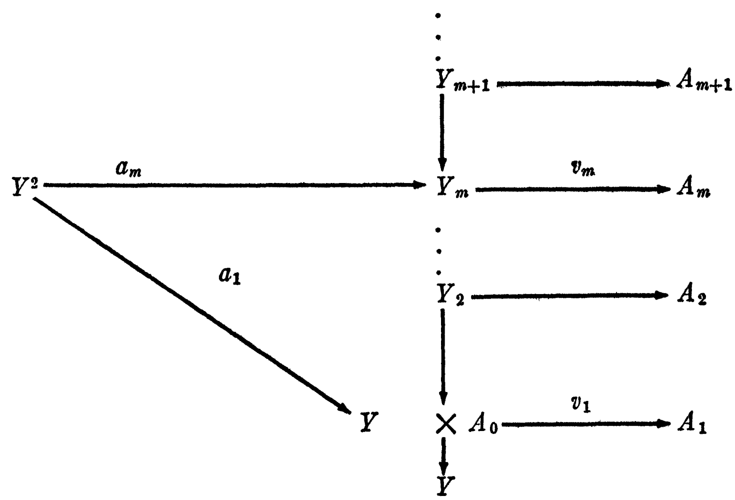
Notes. (1) $H^*(Y^2, Y)$ can be easily computed as an $A(Y)$ module in terms of $H^*(Y)$ by the results of [5].

(2) Low level computations with the spectral sequence are not difficult. However, the results can be obtained also, and sometimes more easily, by the methods of [6], [7]. The spectral sequence should ultimately prove valuable for proving general theorems about $[X, Y]_B$ (e.g., about immersion groups).

(3) If $B = *$ (a point) then the spectral sequence reduces, after a little manipulation of E_2, to the Adams spectral sequence for $[X, Y]$.

2. **Sketch of the proof.** Let $\mathfrak{J}Y$ be the category of all triples $(Z, \check{z}, \hat{z})$ where $Y \xrightarrow{\check{z}} Z \xrightarrow{\hat{z}} Y$ and $\hat{z}\check{z} = 1$, i.e., of all coretractions of Y with given retraction. A morphism in the category is a map $m\colon Z \to W$ such that $m\check{z} = \check{w}$ and $\hat{w}m = \hat{z}$. Recall from [6], [7] that one can define a notion of homotopy in $\mathfrak{J}Y$ (in the obvious way) and also cone, suspension, path, and loop functors enjoying the same properties as the usual functors on $\mathfrak{J}*$ (= the ordinary category of pointed spaces and maps). The cone-suspension sequence (Puppe sequence) and the path-loop sequence are exact after application of $\langle -, Z\rangle$ and $\langle Z, -\rangle$ respectively. $\langle -, -\rangle$ denotes the set of homotopy classes of maps in the category. In brief, all the notions concerning $\mathfrak{J}*$ generalize to $\mathfrak{J}Y$.

We will now apply an upside down version of Adams' method [1] to $q\colon Y^2 \to Y$. Since $[X, Y]_B = [X, Y^2]_Y = \langle X \vee Y, Y^2\rangle$, we can work in $\mathfrak{J}Y$. Suspension of Y^2 in $\mathfrak{J}Y$ has the effect of suspending F in $\mathfrak{J}*$. Successively larger pieces of the spectral sequence are obtained by taking successively higher suspensions of Y^2. We will be content here with one piece. Assume $\mathrm{conn}(F) = n$. Consider the following commutative diagram in $\mathfrak{J}*$.



Each A_i is a product of $K(Z_p, j)$'s. $a_1 = (q, u)$, where $u = (u_1, u_2, \cdots)$ and the i^*u_j's form a set of A generators for $H^j(F)$, $j \leqq 2n+1$. $v_m = (v_{m,1}, v_{m,2}, \cdots)$ and the $v_{m,j}$'s form a set of $A(Y)$ generators for $(\ker a_m^*)^j$, $j = 2n+1$.

The tower can be formally written as a new tower in $\mathfrak{J} Y$ simply by replacing A_m, $m > 0$, by $Y \times A_m$ and v_m by (q_m, v_m) where q_m: $Y_m \to Y$ is from the original tower. Each fibration $Y_m \to Y_{m-1}$ is a fibration in $\mathfrak{J} Y$ induced from a principal fibration in $\mathfrak{J} Y$.

Now apply the functor $\langle X \vee Y, -\rangle$. The resulting exact couple gives the promised piece of the sequence.

REFERENCES

1. J. F. Adams, *On the structure and applications of the Steenrod algebra,* Comment. Math. Helv. **32** (1958), 180–214.

2. J. C. Becker, *Homotopy theory of cross-sections and equivariant maps in the stable range,* Thesis, University of Michigan, 1964.

3. ———, *Cohomology and the classification of liftings* (to appear).

4. A. Liulevicius, *The cohomology of Massey-Peterson algebra* (to appear).

5. W. Massey and F. Peterson, *The cohomology structure of certain fiber spaces.* I. Topology **4** (1964), 47–66.

6. J. F. McClendon, *Higher order twisted cohomology operations,* Thesis, Univ. of California, Berkeley, 1966.

7. ———, *Higher order twisted cohomology operations* (to appear).

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