# GENERATING GROUPS OF NILPOTENT VARIETIES

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Communicated by Walter Feit, March 21, 1968

Problem 14 of Hanna Neumann's book [3] asks for a proof of a conjecture which is contradicted by the following

THEOREM. *If $c$ is an integer greater than 2, then the variety $\mathfrak{N}_c$ of all nilpotent groups of class at most $c$ is generated by its free group $F_{c-1}(\mathfrak{N}_c)$ of rank $c-1$ but not by its free group $F_{c-2}(\mathfrak{N}_c)$ of rank $c-2$.*

In the terms of [3], this means that for $c>2$ one has $d(c)=c-1$ rather than $d(c)=[c/2]+1$ as suggested in Problem 14; correspondingly, [3, 35.35] is false for $c=5$ and 6. (Professor Neumann has confirmed that her proofs were faulty.)

The theorem was suggested by Graham Higman's approach to nilpotent varieties of class $c$ and prime exponent greater than $c$, via the representation theory of the general linear groups [1]. In particular, he remarked that each critical group in such a variety can be generated by $c-1$ elements (if $c>2$). Since $F_c(\mathfrak{N}_c)$ generates $\mathfrak{N}_c$ (cf. [3, 35.12]) and is residually of prime exponent (cf. Higman [2]), it follows easily that $F_{c-1}(\mathfrak{N}_c)$ generates $\mathfrak{N}_c$. It is not difficult to use Higman's method for confirming the second half of the theorem as well.

In this note we outline a proof which avoids the conceptual complexity of Higman's approach; the price of this is paid for in length. Unless otherwise specified, our notation and terminology follow Hanna Neumann's book [3].

To prove the first half of the theorem, it is sufficient to find a set of homomorphisms from $F_c(\mathfrak{N}_c)$ to $F_{c-1}(\mathfrak{N}_c)$ whose kernels intersect trivially. Hanna Neumann did just this in the proof of [3, 35.35] for $c=4$, and the same idea works generally: if $\{a_1, \cdots, a_c\}$ is a free generating set for $F_c(\mathfrak{N}_c)$ and $\{b_1, \cdots, b_{c-1}\}$ is one for $F_{c-1}(\mathfrak{N}_c)$, then the $2c-1$ homomorphisms $\delta_1, \cdots, \delta_c, \theta_1, \cdots, \theta_{c-1}$ defined by

$$\begin{aligned} a_j\delta_i &= b_j \quad \text{if } j < i, & \text{and} \quad a_j\theta_i &= b_j \quad \text{if } j \leqq i,\\ a_j\delta_i &= 1 \quad \text{if } j = i, & a_j\theta_i &= b_{j-1} \quad \text{if } j > i\\ a_j\delta_i &= b_{j-1} \quad \text{if } j > i & & \end{aligned}$$

will do. The verification of this makes use of the unique representation of the elements of $F_c(\mathfrak{N}_c)$ in terms of basic commutators in $\{a_1, \cdots, a_c\}$ as defined by Martin Ward [4]. The case of odd $c$ is comparatively straightforward, that of even $c$ requires lengthy and careful discussion.

For the second half, we show that the word $w_c$ defined below is a law in $F_{c-2}(\mathfrak{N}_c)$, and then we exhibit a nilpotent group $G_c$ of class $c$ (and soluble length 3) in which $w_c$ is not a law. The choice of this word was suggested by Higman's approach. Let $S$ denote the group of all permutations of $\{1, \cdots, c-1\}$, and let $\epsilon(\sigma)$ be 1 or $-1$ according as $\sigma$ is an even or odd permutation; then

$$w_c = \prod_{\sigma \in S} [x_c, x_{(c-1)\sigma}, \cdots, x_{1\sigma}]^{\epsilon(\sigma)}.$$

If $F = F_{c-2}(\mathfrak{N}_c)$ is freely generated by $\{a_1, \cdots, a_{c-2}\}$, then every element of $F$ can be written in the form

$$d \prod_{j=1}^{c-2} a_j^{\alpha(j)}$$

where $d$ belongs to the commutator subgroup $F'$ of $F$. Substituting $d_i \prod_{j=1}^{c-2} a_j^{\alpha(i,j)}$ for $x_i$ in $w_c$, one finds that every value of $w_c$ in $F$ is of the form

$$\prod_{\phi} [a_{c\phi}, a_{(c-1)\phi}, \cdots, a_{1\phi}]^{\beta(\phi)}$$

where $\phi$ runs through all maps from $\{1, \cdots, c\}$ to $\{1, \cdots, c-2\}$ and

$$\beta(\phi) = \alpha(c, c\phi) \sum_{\sigma \in S} \epsilon(\sigma) \prod_{i=1}^{c-1} \alpha(i\sigma, i\phi).$$

Now two of $1\phi, \cdots, (c-1)\phi$ must be equal: say, $r\phi = s\phi$. Let $T$ be a transversal of the subgroup of $S$ generated by the transposition $(rs)$ such that every element of $S$ not in $T$ can be written uniquely as $(rs)\tau$ with $\tau \in T$. Then

$$\begin{aligned}\beta(\phi) &= \alpha(c, c\phi) \sum_{\tau \in T} \epsilon(\tau) \left\{\prod_{i=1}^{c-1} \alpha(i\tau, i\phi) - \prod_{i=1}^{c-1} \alpha(i(rs)\tau, i\phi)\right\} \\ &= 0 \quad \text{because } r\phi = s\phi.\end{aligned}$$

Thus $w_c$ is a law in $F$.

The choice of $G_c$ was influenced by the observation that $w_c$ can be rewritten as follows. Put $n = [\frac{1}{2}(c+1)]$ so that $c = 2n$ or $2n-1$; note that $n > 1$ as $c > 2$. For each $j$ in $\{1, \cdots, n\}$, let $K_j$ be the subgroup of $S$ generated by the transpositions $(2i-1\ 2i)$ with $i \in \{j, \cdots, n-1\}$: thus $K_n$ is the identity subgroup of $S$. Let $T$ denote an arbitrary transversal of $K_1$ in $S$ so that $S = K_1 T$. Then $w_c = w_c^T d_c^T$ where

$$w_{2n}^{T} = \prod_{\tau\in T} [[x_{2n}, x_{(2n-1)\tau}], [x_{(2n-2)\tau}, x_{(2n-3)\tau}], \cdots, [x_{2\tau}, x_{1\tau}]]^{\epsilon(\tau)},$$

$$w_{2n-1}^{T} = \prod_{\tau\in T} [x_{2n-1}, [x_{(2n-2)\tau}, x_{(2n-3)\tau}], \cdots, [x_{2\tau}, x_{1\tau}]]^{\epsilon(\tau)},$$

and $d_c^T$ is a law in $\mathfrak{N}_c$. This is a consequence of the fact that

$$[y, [x_{2n-2}, x_{2n-3}], \cdots, [x_2, x_1]]$$
$$= d_j \prod_{\sigma\in K_j} [y, x_{(2n-2)\sigma}, \cdots, x_{(2j-1)\sigma}, [x_{(2j-2)\sigma}, x_{(2j-3)\sigma}], \cdots, [x_{2\sigma}, x_{1\sigma}]]^{\epsilon(\sigma)}$$

where $d_j$ is a law in $\mathfrak{N}_c$: a tautology if $j=n$ and $d_n=1$, and proved in general by reverse induction on $j$.

The group $G_c$ is constructed as follows. Let $W_n=F_{2n}(\mathfrak{N}_n\cap\mathfrak{A}^2)$ be freely generated by $C=\{u_2, u_4, \cdots, u_{2n}, v_1, \cdots, v_n\}$. For each $i$ in $\{1, \cdots, n\}$, there is an automorphism $\gamma_i$ of $W_n$ which maps $u_{2i}$ to $u_{2i}v_i$ and fixes all other elements of $C$. It is easy to check that $\gamma_1, \cdots, \gamma_n$ generate a free abelian subgroup of rank $n$ in the automorphism group of $W_n$. The group $H_n$ is the splitting extension of $W_n$ by a free abelian group of rank $n$ freely generated by $u_1, u_3, \cdots, u_{2n-1}$ where the homomorphism specifying the extension maps $u_{2i-1}$ to $\gamma_i$ for all $i$. We take $G_c$ to be the largest nilpotent-of-class-$c$ factor group of the subgroup of $H_n$ generated by $\{u_1, \cdots, u_c\}$.

We now sketch the verification that this $G_c$ has the required properties. If $z_1, \cdots, z_r\in C$, then $[z_1, \cdots, z_r, u_{2i-1}]$ is a (possibly empty) product of left-normed commutators of weight at least $r$ with entries from $C$ in which all the commutators of weight $r$ have fewer entries $u_{2i}$ than $[z_1, \cdots, z_r]$. From this, a routine argument shows that every left-normed commutator of weight $2n$ with entries from $\{u_1, \cdots, u_{2n}\}$ lies in the $n$th term of the lower central series of the subgroup of $W_n$ generated by $\{v_1, \cdots, v_n\}$. It follows that $H_n$ has class $2n$.

Suppose that $c=2n$, so that $G_c=H_n$, and evaluate $w_{2n}$ when $u_i$ is substituted for $x_i$. Since each $d_{2n}^T$ is a law in $H_n$, we get the same value from substituting in any $w_{2n}^T$, and we exploit our freedom in choosing $T$. Let $R$ be the group of all permutations on $\{1, \cdots, n-1\}$; define a monomorphism $\rho\to\rho^*$ from $R$ to $S$ by $(2k-1)\rho^*=2(k\rho)-1$, $(2k)\rho^*=2(k\rho)$ if $k<n$, and $(2n-1)\rho^*=2n-1$, and denote its image by $R^*$. As $R^*$ avoids $K_1$, it can be extended to a transversal $T$. Observe that $[u_i, u_j]$ lies in the commutator subgroup of $W_n$ unless $|i-j|=1$ and $\max(i, j)$ is even; hence

$$[[u_{2n}, u_{(2n-1)\tau}], [u_{(2n-2)\tau}, u_{(2n-3)\tau}], \cdots, [u_{2\tau}, u_{1\tau}]] = 1$$

unless $\tau \in K_1R^*$; and $K_1R^* \cap T = R^*$. Finally, note that every element of $R^*$ is even. Using this information, we find that

$$\begin{aligned}\prod_{\sigma \in S} [u_{2n}, u_{(2n-1)\sigma}, \cdots, u_{1\sigma}]^{\epsilon(\sigma)} &= \prod_{\rho^* \in R^*} [[u_{2n}, u_{(2n-1)\rho^*}], \cdots, [u_{2\rho^*}, u_{1\rho^*}]] \\ &= \prod_{\rho \in R} [v_n, v_{(n-1)\rho}, \cdots, v_{1\rho}] \\ &= \prod_{r=1}^{n-1} [v_n, v_r, v_1, \cdots, v_{r-1}, v_{r+1}, \cdots, v_{n-1}]^{(n-2)!},\end{aligned}$$

where the last equality holds because $W_n$ is metabelian. By [**3**, 36.32], this value of $w_{2n}$ is not trivial. This proves that if $c$ is even, $w_c$ is not a law in $G_c$.

A similar argument shows that if $c = 2n-1$ and $u_i$ is substituted for $x_i$, the value of $w_{2n-1}$ is

$$\prod_{r=1}^{n-1} [u_{2n-1}, v_r, v_1, \cdots, v_{r-1}, v_{r+1}, \cdots, v_{n-1}]^{(n-2)!}$$

which does not belong to the $2n$th term of the lower central series of $H_n$. Thus also when $c$ is odd, $w_c$ is not a law in $G_c$, and this completes the outline of the proof.

REMARK (*added in proof* July 9, 1968). An alternative and independent proof of this result has been obtained by Professor F. Levin and submitted to J. Austral. Math. Soc.

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