NOTE ON PRINCIPAL S^3-BUNDLES

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In this note we construct two principal S^3-bundles whose total spaces E_α, E_β are closed smooth manifolds having the properties

(i) E_α, E_β are of different homotopy types, $E_\alpha \not\simeq E_\beta$;

(ii) $E_\alpha \times S^3$, $E_\beta \times S^3$ are diffeomorphic.

The method of construction is a modified dual of that employed in [1] to demonstrate the failure of wedge-cancellation.

Let $a, b \in \pi_n(S^3)$, let B be the classifying space for S^3, and let $\alpha, \beta \in \pi_{n+1}(B)$ be the elements corresponding to a, b respectively. Let $\pi_\alpha: E_\alpha \to S^{n+1}$, $\pi_\beta: E_\beta \to S^{n+1}$ be the bundle projections induced by[1] α, β.

THEOREM 1. *$E_\alpha \simeq E_\beta$ if and only if $\beta = \pm\alpha$ (equivalently, $b = \pm a$).*

PROOF. Sufficiency is obvious, so we suppose $E_\alpha \simeq E_\beta$ and seek to prove $\beta = \pm\alpha$. If $n \leqq 2$, the assertion is trivial. Now there are cell-decompositions

$$E_\alpha = S^3 \cup_a e^{n+1} \cup e^{n+4}, \qquad E_\beta = S^3 \cup_b e^{n+1} \cup e^{n+4}.$$

Thus if $n=3$, a and b are integers and $H_3(E_\alpha) = Z_{|a|}$, $H_3(E_\beta) = Z_{|b|}$, whence $|a| = |b|$. We assume now that $n \geqq 4$ and let $h: E_\alpha \simeq E_\beta$. We may suppose $h(S^3) \subseteq S^3$ and then $h|S^3$ is of degree ± 1. From the exact homotopy sequence we infer that h induces an isomorphism $\pi_{n+1}(E_\alpha, S^3) \cong \pi_{n+1}(E_\beta, S^3)$; these groups are cyclic infinite, generated by i_α, i_β say, so that $h_*(i_\alpha) = \pm i_\beta$. We have a commutative square

$$\begin{array}{ccc} \pi_{n+1}(E_\alpha, S^3) & \overset{h_*}{\cong} & \pi_{n+1}(E_\beta, S^3) \\ \downarrow \partial & & \downarrow \partial \\ \pi_n(S^3) & \cong & \pi_n(S^3) \end{array}$$

where the bottom isomorphism is multiplication by ± 1, $\partial(i_\alpha) = a$, $\partial(i_\beta) = b$. Thus $\pm b = \pm a$ or $\beta = \pm\alpha$.

Let $E_{\alpha\beta} \to E_\alpha$ be induced from π_β by $\pi_\alpha: E_\alpha \to S^{n+1}$, and let $E_{\beta\alpha} \to E_\beta$ be defined similarly.

THEOREM 2. *$E_{\alpha\beta} = E_{\beta\alpha}$. Moreover, $E_{\alpha\beta}$ is equivalent to $E_\alpha \times S^3$ if $\beta \circ \pi_\alpha = 0$ and $E_{\beta\alpha}$ is equivalent to $E_\beta \times S^3$ if $\alpha \circ \pi_\beta = 0$.*

[1] Here and later we deliberately confuse maps and homotopy classes.

Thus it remains to choose α,β so that $\beta\circ\pi_\alpha=0$, $\alpha\circ\pi_\beta=0$, and $\beta\neq\pm\alpha$. We now assume $n\geqq 4$.

We construct the Puppe sequence for the inclusion $S^3\xrightarrow{i}E_\alpha$, namely,

$$S^3\xrightarrow{i}E_\alpha\xrightarrow{p}Y\xrightarrow{u}S^4\rightarrow\cdots,$$

where $Y=S^{n+1}\cup e^{n+4}$. Plainly $\pi_\alpha=q\circ p$ for $q\colon Y\rightarrow S^{n+1}$, so that

$$0=\alpha\circ\pi_\alpha=\alpha\circ q\circ p,$$

whence

$$\alpha\circ q=d\circ u,\quad\text{for some } d\colon S^4\rightarrow B.$$

We have the 'fibration'

$$S^7\xrightarrow{h}S^4\xrightarrow{e}B,$$

where h is the Hopf map and e generates $\pi_4(B)\cong Z$. Then, since Y is a double suspension,

$$u=h\circ v+u',\quad\text{for some } v\colon Y\rightarrow S^7,$$

where u' is a suspension. Moreover, $d=me$ for some integer m and, for any integer s,

$$se\circ h=\frac{s(s-1)}{2}[e,e],$$

where [,] denotes the Whitehead product. Thus, for any integer l,

$$\begin{aligned}l\alpha\circ q&=l(\alpha\circ q),\quad\text{since } q \text{ is a suspension}\\&=l(d\circ u)\\&=l(d\circ h\circ v+d\circ u')\\&=d\circ h\circ lv+ld\circ u',\quad\text{since } u' \text{ is a suspension.}\end{aligned}$$

On the other hand

$$\begin{aligned}ld\circ u&=lme\circ h\circ v+ld\circ u'\\&=\frac{lm(lm-1)}{2}[e,e]\circ v+ld\circ u'.\end{aligned}$$

Now 12 $[e,e]=0$. Thus if we choose l so that $lv=0$ and $l\equiv 0\pmod{24}$, then

$$l\alpha\circ q=ld\circ u'=ld\circ u.$$

But then $l\alpha \circ \pi_\alpha = l\alpha \circ q \circ p = 0$. Now we have an exact sequence

$$\pi_{n+4}(S^7) \rightarrow \pi(Y, S^7) \rightarrow \pi_{n+1}(S^7);$$

thus if r_1 is the exponent of $\pi_{n+4}(S^7)$ and r_2 is the exponent of $\pi_{n+1}(S^7)$ we may take

$$l_0 = \text{l.c.m.}(r_1 r_2, 24)$$

and we have

THEOREM 3. *If* $l_0 | l$ *and* $\beta = l\alpha$, *then* $\beta \circ \pi_\alpha = 0$.

Naturally we may interchange the roles of α,β here; l_0 remains unchanged. We take $n = 17$; then $\pi_{17}(S^3) = Z_{30}$ (see [2]) and we choose $a \in \pi_{17}(S^3)$ of order 5. From [2] we see that $r_1 = 24$, $r_2 = 504$, so we may certainly choose l so that $l_0 | l$ and $l \equiv 2 \bmod 5$. Thus if $\beta = 2\alpha$, $\beta \circ \pi_\alpha = 0$. But then $b = 2a$ is of order 5 and $a = 3b$, and we may choose l so that $l_0 | l$ and $l \equiv 3 \bmod 5$. Thus we also have $\alpha \circ \pi_\beta = 0$. On the other hand $\beta \neq \pm\alpha$, so that we have constructed the promised example, in which E_α, E_β are principal S^3-bundles over S^{18}.

REFERENCES

1. Peter Hilton, *On the Grothendieck group of compact polyhedra*, Fund. Math. **61** (1967), 199–214.

2. Hirosi Toda, *Composition methods in homotopy groups of spheres*, Ann. of Math. Studies, no. 49, Princeton Univ. Press, Princeton, N. J., 1962.

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