CONSTRUCTING 3-MANIFOLDS FROM GROUP HOMOMORPHISMS

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1. **Introduction.** Let S be a closed, orientable 2-manifold of genus $n>0$. Let F_1 and F_2 be free groups of rank n and denote by $F_1 \times F_2$ their direct product. Fix a point s_0 of S and suppose η_1, η_2 are homomorphisms of $\pi_1(S, s_0)$ *onto* F_1 and F_2 respectively. The homomorphism

$$\eta_1 \times \eta_2 \colon \pi_1(S, s_0) \to F_1 \times F_2$$

is called a *splitting homomorphism of* $\pi_1(S, s_0)$. Let M be a closed, orientable 3-manifold. In [**3**] J. Stallings introduced a natural splitting homomorphism induced by a Heegaard splitting of M. The purpose of this paper is to announce that for any splitting homomorphism there is a closed, orientable 3-manifold M and a Heegaard splitting of M so that the induced splitting homomorphism is equivalent to the given splitting homomorphism. This is Theorem 4.1 of §4. See §2 for definitions.

It is shown in Theorem 4.2 that two conjectures made by J. Stallings in [**3**] are true if and only if Poincaré's Conjecture that any closed, simply-connected 3-manifold is a 3-sphere, is true. These conjectures appear in §4 as Conjecture B and Conjecture D (using the notation of [**3**]).

Perhaps of independent interest is the Corollary to Lemma 3.2 of §3. It states that there is a homomorphism of the fundamental group of a closed, orientable surface of genus n *onto* a free group of rank k iff $k \leq n$.

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2. **Notation and definitions.** The term *map* is used to mean continuous function. If f is a map from (S, s_0) to (X, x_0), then the homomorphism of $\pi_1(S, s_0)$ to $\pi_1(X, x_0)$ induced by f is denoted f_*. Suppose l is a map of S^1 into a pathwise connected space S. Then l

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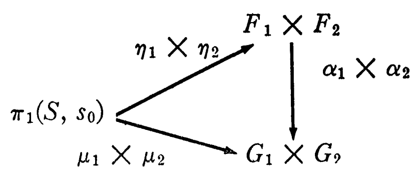
defines a *conjugate class* of elements of $\pi_1(S, s_0)$. This class is denoted by $\{l\}$. For any normal subgroup N of $\pi_1(S, s_0)$, the statement $\{l\} \in N(\{l\} \notin N)$ is well defined.

A 3-manifold-with-boundary is called a *cube-with-handles of genus* $n \geqq 0$ if it is orientable and a regular neighborhood of a finite, connected graph with Euler characteristic $1-n$. Let M be a closed, orientable 3-manifold. It is known that $M = U \cup V$ where U and V are cubes-with-handles and $U \cap V$ is their common boundary. The pair (U, V) is called a *Heegaard splitting* of M. The *genus* of (U, V) is defined to be the genus of $S = \text{Bd } U = \text{Bd } V$. Choose $s_0 \in S$. The inclusion maps of S into U and V induce homomorphisms u_* and v_* of $\pi_1(S, s_0)$ *onto* $\pi_1(U, s_0)$ and $\pi_1(V, s_0)$ respectively. The homomorphism

$$u_* \times v_*: \pi_1(S, s_0) \to \pi_1(U, s_0) \times \pi_1(V, s_0)$$

is called the *splitting homomorphism of* $\pi_1(S, s_0)$ *induced by* (U, V).

Suppose $\eta_1 \times \eta_2$: $\pi_1(S, s_0) \to F_1 \times F_2$ and $\mu_1 \times \mu_2$: $\pi_1(S, s_0) \to G_1 \times G_2$ are splitting homomorphisms of $\pi_1(S, s_0)$. Then $\eta_1 \times \eta_2$ is said to be *equivalent* to $\mu_1 \times \mu_2$ if there are isomorphisms α_1 of F_1 onto G_1 and α_2 of F_2 onto G_2 so that the diagram



commutes.

3. Mapping surfaces into wedges. The details of the proofs for the lemmas of this section will appear elsewhere. The next few paragraphs give notation which will be used for these lemmas.

Suppose f is a piecewise linear (PL) map of the closed, orientable 2-manifold S into T, a wedge at t_0 of k simple closed curves $X_1, \cdots, X_k$. Suppose $f(s_0) = t_0$. For each i, $1 \leqq i \leqq k$, choose $x_i \in X_i - \{t_0\}$ which is not a vertex in the subdivision of T for which f is simplicial. Then each component of $f^{-1}(x_i)$ is a polyhedral simple closed curve in S. Let J_1 and J_2 be distinct components of $f^{-1}(x_j)$. Suppose that A is a polyhedral arc in $S - \{s_0\}$ from J_1 to J_2 such that $A \cap \bigcup_{i=1}^{k} f^{-1}(x_i) = \text{Bd } A$.

Let $Q_1 \subset J_1$ and $Q_2 \subset J_2$ be arbitrarily small arcs each containing an endpoint of A in its interior. Let Q denote a small regular neighborhood of A which meets $\bigcup_{i=1}^{k} f^{-1}(x_i)$ in Q_1 and Q_2 each of which is contained in Bd Q. Then the closure of Bd $Q - (Q_1 \cup Q_2)$ is two disjoint arcs A_1 and A_2. Both A_1 and A_2 are arcs from J_1 to J_2.

Let I denote the unit interval and let $U(x_i)$ denote an arc in $X_i - t_0$ with x_i in its interior. Then $U(x_i) = x_i \times I$ has a product structure with $x_i = x_i \times \frac{1}{2}$. Let J be a polyhedral simple closed curve in S. Then a neighborhood, $U(J)$, of J is said to have a *product structure* if $U(J) = J \times I$ with $J = J \times \frac{1}{2}$. A map f of S into T will be called *transverse with respect to* $\bigcup_{i=1}^{k} \{x_i\}$ if each component of $f^{-1}(x_i)$ is a simple closed curve, there exists $U(x_i)$ as above such that each component of $f^{-1}(U(x_i))$ has a product structure, and f maps each fiber of $f^{-1}(U(x_i))$ homeomorphically onto a fiber of $U(x_i)$.

LEMMA 3.1. *If f maps the arc A into the trivial loop of T based at x_j, then there is a* PL *map g of S into T so that*

(i) *g is homotopic to f keeping s_0 fixed,*

(ii) $g^{-1}(x_i) = f^{-1}(x_i)$, $i \neq j$,

(iii) $g^{-1}(x_j) = f^{-1}(x_j) \cup (A_1 \cup A_2) - (Q_1 \cup Q_2)$, *and*

(iv) *g is transverse with respect to* $\bigcup_{i=1}^{k} \{x_i\}$.

LEMMA 3.2. *Let S be a closed, orientable 2-manifold of genus $n > 0$. Suppose f is a map of S into T, a wedge of k simple closed curves $X_1, \cdots, X_k$ at t_0 and $f(s_0) = t_0$. If f_* is an epimorphism, then there is a* PL *map g of S into T so that*

(i) *g is homotopic to f keeping s_0 fixed,*

(ii) *for each i, $1 \leqq i \leqq k$, there is a point $x_i \in X_i - \{t_0\}$ so that $g^{-1}(x_i)$ is a single simple closed curve J_i in S,*

(iii) *$S - \bigcup_{i=1}^{k} J_i$ is connected, and*

(iv) *g is transverse with respect to* $\bigcup_{i=1}^{k} \{x_i\}$.

The proof of Lemma 3.2 uses the notion of "binding-tie" introduced by J. Stallings in [2]. Although, here a more delicate argument is required since the domain of f is the 2-manifold S.

An interesting corollary of Lemma 3.2 is

COROLLARY 3.3. *There is a homomorphism of the fundamental group of a closed, orientable surface of genus n onto a free group of rank k if and only if $k \leqq n$.*

LEMMA 3.4. *Let S be a closed, orientable surface of genus $n > 0$. Suppose f is a map of S into T, a wedge of n simple closed curves at t_0 and $f(s_0) = t_0$. If f_* is an epimorphism, then there is a* PL *map g of S into T so that*

(i) *g is homotopic to f keeping s_0 fixed, and*

(ii) *the mapping cylinder of g is topologically equivalent to a cube-with-handles of genus n.*

4. Constructing three-manifolds.

THEOREM 4.1. *Let S be a closed, orientable 2-manifold of genus $n>0$. Suppose $\eta_1\times\eta_2$ is a splitting homomorphism of $\pi_1(S, s_0)$ into $F_1\times F_2$. Then there is a closed, orientable 3-manifold M and a Heegaard splitting (U, V) of M so that the splitting homomorphism induced by (U, V) is equivalent to $\eta_1\times\eta_2$.*

PROOF. For $k=1, 2$, let T_k be a wedge at t_k of n simple closed curves. Identify $\pi_1(T_k, t_k)$ with F_k and let f_k be a map of (S, s_0) into (T_k, t_k) so that $(f_k)_*=\eta_k$. From Lemma 3.4 and for $k=1, 2$, there is a PL map g_k of (S, s_0) into (T_k, t_k) so that g_k is homotopic to f_k keeping s_0 fixed and the mapping cylinder $C(g_k)$ of g_k is a cube-with-handles of genus n.

Let M be the closed three-manifold obtained by considering $C(g_1)\cup C(g_2)$ as a disjoint union and identifying $s\in S=\text{Bd } C(g_1)$ with $s\in S=\text{Bd } C(g_2)$.

There is an isomorphism α_k of $\pi_1(C(g_k), s_0)$ onto $\pi_1(T_k, t_k)$ so that if e_k is the inclusion of S into $C(g_k)$, then $\alpha_k(e_k)_*=(f_k)_*$. It follows that the diagram

$$\begin{array}{ccc} & \overset{(f_1)_*\times(f_2)_*}{\nearrow} & \pi_1(T_1, t_1)\times\pi_1(T_2, t_2) \\ \pi_1(S, s_0) & & \uparrow \alpha_1\times\alpha_2 \\ & \underset{(e_1)_*\times(e_2)_*}{\searrow} & \pi_1(C(g_1), s_0)\times\pi_1(C(g_2), s_0) \end{array}$$

commutes. Let $U=C(g_1)$ and $V=C(g_2)$. The splitting homomorphism induced by (U, V) is $(e_1)_*\times(e_2)_*$. Since $(f_1)_*\times(f_2)_*=\eta_1\times\eta_2$ the theorem is proved.

The notation $G_1 * G_2$ is used for the free product of the groups G_1 and G_2 ([1, Vol. II, p. 11]).

In [3], J. Stallings states the following conjectures:

CONJECTURE B. *Let S be a closed, orientable 2-manifold of genus $n>1$. Let F_1 and F_2 be free groups of rank n. Let η: $\pi_1(S)\rightarrow F_1\times F_2$ be an epimorphism. Then there is a nontrivial element $\{l\}\in$kernel η where l is a simple closed curve on S.*

CONJECTURE D. *In the situation of Conjecture* B, *the map η: $\pi_1(S)\rightarrow F_1\times F_2$ can be factored through an essential map of $\pi_1(S)$ into some free product $G_1 * G_2$.*

A homomorphism ϕ of a group G into a free product $G_1 * G_2$ is *essential* if there is no element $g\in G_1 * G_2$ such that $g\phi(G)g^{-1}$ is contained in one of the factors G_1 or G_2.

THEOREM 4.2. *Conjecture* B *is true if and only if Poincaré's Conjecture is true if and only if Conjecture* D *is true.*

PROOF. It is shown in [3] that Conjecture B implies Poincaré's Conjecture.

It is now shown that Poincaré's Conjecture implies Conjecture D. Let $\eta=\eta_1\times\eta_2$ be the epimorphism given in Conjecture D. Let M denote the 3-manifold with Heegaard splitting (U, V) guaranteed in Theorem 4.1 corresponding to the splitting homomorphism $\eta_1\times\eta_2$ of $\pi_1(S, s_0)$ into $F_1\times F_2$. Then M is simply connected, since η is an epimorphism (Theorem 1 of [3]). Thus by Poincaré's Conjecture, M is the 3-sphere S^3. The Heegaard splitting (U, V) of S^3 has genus $n>1$. By Waldhausen [4], there is a simple closed curve l in S, $\{l\}\neq 1$ in $\pi_1(S)$. Also, if $u_*\times v_*$ is the splitting homomorphism induced by (U, V), then $\{l\}\in\text{kernel } u_*\cap\text{kernel } v_*$, and l separates S. To factor η through a free product, a wedge S' is formed from S by identifying the simple closed curve l to a point. The fundamental group of S' is the desired nontrivial free product.

From Theorem 2 of [3], Conjecture D implies Conjecture B.

REMARK. The author has shown, with appropriate definitions for equivalence, that there is a one-one correspondence between equivalence classes of splitting-homomorphisms and equivalence classes of Heegaard splittings.

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