ON THE ADJOINT OF THE PRODUCT OF OPERATORS

BY SAMUEL S. HOLLAND, JR.

Communicated by P. R. Halmos, April 15, 1968

It is known that if S and T are closed domain-dense linear operators on Hilbert space H, then $(TS)^* \supseteq S^*T^*$. The question, "when does equality obtain?" is an important question, and the only general answers that seem to be known are these two: The well-known theorem that if T is bounded and everywhere defined, then $(TS)^* = S^*T^*$ [1, p. 1189], and von Neumann's important theorem that asserts equality when $S = T^*$ [1, p. 1245]. While von Neumann's result applies to nonbounded S's, it is proved by methods which seem to use in an essential way the close relationship between T and T^*, methods which have not yet yielded information about other S's.

A recent result of William Stenger bears on the question as to when $(TS)^* = S^*T^*$. Stenger proves in [2] that if T is selfadjoint and Y is a projection on a closed subspace of finite codimension, then YTY is selfadjoint. If we knew that $(TY)^* = YT^*$ held under the same hypotheses, we could derive Stenger's theorem as an easy consequence. Because, Y being bounded, we would know that TY was closed, and, since $(TY)^* = YT^*$, also that TY was densely defined (since it would have a single-valued adjoint). Then by the theorem cited in the first paragraph above, we could get

$$(YTY)^* = (Y(TY))^* = (TY)^*Y = YT^*Y$$

which is Stenger's theorem, since in this case $T = T^*$.

The speculative proposition $(TY)^* = TY^*$ is indeed true, and can moreover be generalized so as to provide another reasonably satisfactory answer to the question as to when $(TS)^* = S^*T^*$. In this paper I adapt Stenger's ideas to prove this result:

THEOREM. *If T is a closed domain-dense linear operator on a Hilbert space H, and S is a bounded everywhere-defined linear operator whose image is a closed subspace of finite codimension in H, then $(TS)^* = S^*T^*$.*

Obviously, the case above is covered by the theorem when we set $S = Y$, Y a projection on a closed subspace of finite codimension.

We can also recover part of the (false) generalization of his theorem that Stenger refutes in the last paragraph of his paper [2]:

COROLLARY. *If S and T are selfadjoint, and if S is bounded, has a closed image, and has a finite dimensional kernel, then STS is selfadjoint.*

I have based the proof of the theorem on three subsidiary results.

LEMMA 1. *Suppose that the closed subspace X of the Hilbert space H has* $\dim(X^{\perp}) < \infty$. *If the subspace D is dense in H, then the subspace $D \cap X$ is dense in X.*

This result is known [3], [4, p. 103].

The next two lemmas are stated somewhat more generally than necessary for the proof of the theorem, but each has some independent interest in the form given.

LEMMA 2. *Let S be a linear operator on Hilbert space, P the projection on* $\ker(S)^{\perp}$, *Q the projection on the closure of* $\operatorname{im}(S)$. *Then $S^{\#} = PS^{-1}Q$ is a single-valued, densely defined linear operator that satisfies the following conditions:*

$$\operatorname{dom}(S^{\#}) = \operatorname{im}(S) \oplus \operatorname{im}(S)^{\perp}, \tag{1}$$

$$SS^{\#} = Q \textit{ restricted to } \operatorname{dom}(S^{\#}), \tag{2}$$

$$S^{\#}S = P \textit{ restricted to } \operatorname{dom}(S), \tag{3}$$

$$S^{\#} \textit{ is closed when } S \textit{ is closed.} \tag{4}$$

When S is closed, the following statements are also valid:

$$\operatorname{im}(S) \textit{ closed} \Rightarrow S^{\#} \textit{ bounded and everywhere defined;} \tag{5}$$

$$S^{\#} \textit{ bounded} \Rightarrow S^{\#} \textit{ everywhere defined and } \operatorname{im}(S) \textit{ closed.} \tag{6}$$

LEMMA 3. *Let S, T be linear operators on Hilbert space. We have*

$$\operatorname{dom}(TS) = [S^{\#}(\operatorname{dom}(T) \cap \operatorname{im}(S))] \oplus \ker(S).$$

Referring to the main theorem, I have hypothesized the boundedness of S in order to force $\operatorname{dom}(S^*) \supseteq \operatorname{im}(T^*)$, and that seems to be the only consequence of boundedness that is used. Thus there is a reasonable possibility that the assumption that S is bounded can be replaced by a weaker assumption without destroying the validity of the theorem.

A detailed paper has been submitted for publication elsewhere.

REFERENCES

1. N. Dunford and J. T. Schwartz, *Linear operators. Part* II: *Spectral theory*, Interscience, New York, 1963.

2. W. Stenger, *On the projection of a selfadjoint operator*, Bull. Amer. Math. Soc. **74** (1968), 369–372.

3. I. Amemiya and H. Araki, *A remark on Piron's paper*, Publ. Res. Inst. Math. Sci. Ser. A **12** (1966/67), 423–427.

4. S. Goldberg, *Unbounded linear operators: Theory and applications*, McGraw-Hill, New York, 1966.

UNIVERSITY OF MASSACHUSETTS, AMHERST