SERRE SEQUENCES AND CHERN CLASSES

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Let A be a commutative noetherian ring. A theorem of J.-P. Serre asserts that if P is a finitely generated projective A-module whose rank exceeds the Krull dimension of the maximal ideal space of A, then P has A as a direct summand. We denote, for any prescheme X, the set of closed points of X with the induced topology by X_0. Now given $s_1, \cdots, s_n \in P$, let $F(s_1, \cdots, s_n) = \{x \in \text{Spec}(A)_0 : s_1(x), \cdots, s_n(x) \in P/\mathfrak{m}_x P$ are linearly dependent over $A/\mathfrak{m}_x\}$. $F(s_1, \cdots, s_n)$ is a closed subset of Spec $(A)_0$. Implicit in Serre's proof is the stronger result:

THEOREM (SERRE [3]). *Let A be a commutative noetherian ring and let P be a finitely generated projective A-module of rank r. Then there exist $s_1, \cdots, s_r \in P$ such that the codimension of $F(s_p, \cdots, s_r)$ in* Spec $(A)_0$ is $\geqq p$, $1 \leqq p \leqq r$.

This result suggests the following questions.

(A) Let X be a noetherian prescheme and let $\mathcal{E}$ be a locally free coherent $\mathcal{O}_X$-module of rank r. When do there exist global sections $s_1, \cdots, s_r \in \Gamma(X, \mathcal{E})$ such that $\operatorname{codim}_{X_0} F(s_p, \cdots, s_r) \geqq p$, for all p?

A sequence of global sections with these properties will be called a Serre sequence for $\mathcal{E}$. Now, given any sequence $s_1, \cdots, s_r$, of global sections of $\mathcal{E}$, one has the $\mathcal{O}_X$-linear mapping, $(s_p, \cdots, s_r)$: $\mathcal{O}_X^{r-p+1} \to \mathcal{E}$ given locally by $(f_p, \cdots, f_r) \to \sum f_i s_i$. Let $Z_p(s_1, \cdots, s_r)$ be the closed subscheme of X whose structure sheaf is $\operatorname{coker}(\bigwedge^{r-p+1}(s_1, \cdots, s_r)^*)$. We then have a flag, $Z_1(s_1, \cdots, s_r) \supset Z_2(s_1, \cdots, s_r) \supset \cdots \supset Z_r(s_1, \cdots, s_r)$, of subschemes of X, with $Z_p(s_1, \cdots, s_r)_0 = F(s_p, \cdots, s_r)$.

(B) Is the rational equivalence class of $Z_p(s_1, \cdots, s_r)$ independent of the choice of the Serre sequence, $s_1, \cdots, s_r$, for $\mathcal{E}$?

(C) If the answer to (B) is affirmative, what is the significance of these invariants?

Now let A be a fixed commutative noetherian ring, and let X be a prescheme of finite type over A. We say that X is quasi-closed over A if $\pi(X_0) \subset \text{Spec}(A)_0$, π being the structure morphism. We say that A is residually infinite if $A/\mathfrak{m}$ is an infinite field for each maximal ideal $\mathfrak{m}$ of A.

THEOREM 1. *Let A be a commutative noetherian ring and let X be a prescheme of finite type and quasi-closed over A. Given a locally free coherent $\mathcal{O}_X$-module $\mathcal{E}$, spanned by its global sections, then there exists a finite faithfully flat A-algebra B, such that $B \otimes_A \mathcal{E}$ admits a Serre sequence on $B \otimes_A X$. Moreover, we may take $B=A$, if either $X=A$, or A is residually infinite.*

The proof of this theorem is a modified version of H. Bass' argument in the affine case [1]. The Chinese Remainder Theorem, which is the core of the argument in the affine case, is replaced by the existence of rational points in certain Zariski open sets.

COROLLARY 1.1. *Let X be a quasi-projective scheme, quasi-closed over A. Given a locally free coherent $\mathcal{O}_X$-module $\mathcal{E}$, then, after a suitable faithfully flat base change* (cf. Theorem 1), *there is an exact sequence*: $0\to \mathcal{O}_X(-m)^N\to\mathcal{E}\to\mathcal{E}'\to 0$, *where $\mathcal{O}_X(1)$ is a very ample invertible sheaf on X, and $\mathcal{E}'$ is a locally free coherent $\mathcal{O}_X$-module of rank* $\leqq \dim X_0$.

As for (B) and (C), the main tool is the generalized Koszul complex [2]. Let X be a prescheme, and let $\mathcal{E}$ and $\mathcal{F}$ be coherent locally free $\mathcal{O}_X$-modules of ranks m, n respectively. Assume that $m \geqq n$. Given an $\mathcal{O}_X$-linear mapping, $\alpha:\mathcal{E}\to\mathcal{F}$, there is associated to α a canonical chain complex

$$K(\alpha):\cdots\to \sum_{s_i>0} \overset{s_1}{\wedge}\mathcal{F}^*\otimes\overset{s_2}{\wedge}\mathcal{F}^*\otimes\overset{s_3}{\wedge}\mathcal{F}^*\otimes\overset{n+s_1+s_2+s_3}{\wedge}\mathcal{E}$$

$$\to \sum_{s_i>0} \overset{s_1}{\wedge}\mathcal{F}^*\otimes\overset{s_2}{\wedge}\mathcal{F}^*\otimes\overset{n+s_1+s_2}{\wedge}\mathcal{E}\to\sum_{s>0}\overset{s}{\wedge}\mathcal{F}^*\otimes\overset{n+s}{\wedge}\mathcal{E}$$

$$\to \overset{n}{\wedge}\mathcal{E}\xrightarrow{\overset{n}{\wedge}\alpha}\overset{n}{\wedge}\mathcal{F}.$$

Given the global sections $s_1, \cdots, s_r$ of the coherent locally free $\mathcal{O}_X$-module $\mathcal{E}$, we set $K(p;\, s_1, \cdots, s_r)=K((s_p, \cdots, s_r)^*)$ (notation as above). Then we have

THEOREM 2. *Let X be a noetherian Jacobson prescheme of Cohen-Macauley type, and let $\mathcal{E}$ be locally free coherent $\mathcal{O}_X$-module of rank r. A sequence of global sections, $s_1, \cdots, s_r \in \Gamma(X, \mathcal{E})$, is a Serre sequence for $\mathcal{E}$ if and only if $K(p;\, s_1, \cdots, s_r)$ is acyclic, $1\leqq p\leqq r$.*

This is an immediate consequence of an acyclicity criterion developed in [2]. Now let Z_1 and Z_2 be closed subschemes of X. We say that Z_1 and Z_2 are rationally equivalent if there exists a closed subscheme W of $X[t]$, such that

$$W \times_{X[t]} X[0] = Z_1, \qquad W \times_{X[t]} X[1] = Z_2$$

and

$$\mathrm{Tor}_j^{\mathcal{O}_{X[t]}}(\mathcal{O}_W, \mathcal{O}_{X[i]}) = (0) \qquad \text{for } j > 0,\ i = 0, 1.$$

THEOREM 3. *Let X be a noetherian Jacobson prescheme of Cohen-Macauley type, and let $\mathcal{E}$ be a locally free coherent $\mathcal{O}_X$-module of rank r. If $s_1, \cdots, s_r$ and $s_1', \cdots, s_r'$ are Serre sequences for $\mathcal{E}$, then $Z_p(s_1, \cdots, s_r)$ and $Z_p(s_1', \cdots, s_r')$ are rationally equivalent, $1 \leqq p \leqq r$.*

Given a prescheme X, let $K(X)$ be the Grothendieck ring of locally free coherent $\mathcal{O}_X$-modules. Then $K(X)$ is a λ-ring, with augmentation $\epsilon: K(X) \to \mathbf{Z}$ given by the rank. Let $K_p(X)$ be the subgroup of $K(X)$ generated by all elements of the form

$$\gamma^{n_1}(x_1 - \epsilon(x_1))\gamma^{n_2}(x_2 - \epsilon(x_2)) \cdots \gamma^{n_r}(x_r - \epsilon(x_r)),$$

with $\sum n_j \geqq p$, where $\gamma^n(x)$ is the coefficient of t^n in $\gamma_t(x) = \lambda_{t/(1-t)}(x)$. We set $\mathrm{Gr}^\cdot(X) = \bigoplus K_p(X)/K_{p+1}(X)$. Then $\mathrm{Gr}^\cdot(X)$ is a graded ring. If $\mathcal{E}$ is a locally free coherent $\mathcal{O}_X$-module, the pth Chern class of $\mathcal{E}$, with values in $\mathrm{Gr}^\cdot(X)$, is the homogeneous element of degree p represented by $\gamma^p(x - \epsilon(x))$, where x = class of $\mathcal{E}$ (A. Grothendieck).

THEOREM 4. *Let X be as in Theorem 3, and let $\mathcal{E}$ be a locally free coherent $\mathcal{O}_X$-module. If $s_1, \cdots, s_r$ is a Serre sequence for $\mathcal{E}$, then $Z_p(s_1, \cdots, s_r)$ represents a class in $K_p(X)$, and the corresponding class in $\mathrm{Gr}^\cdot(X)$ is the pth Chern class of $\mathcal{E}$.*

The proof is gotten by computing the class of the generalized Koszul complex.

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