EXPECTATIONS IN VON NEUMANN ALGEBRAS

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1. Introduction. Let M be a von Neumann algebra. Let N be a von Neumann subalgebra of M. An expectation from M on N is defined to be a positive map Φ of M into N which preserves the identity and satisfies $\Phi(AX)=A\Phi(X)$ for all X in M and all A in N. Φ is a normal expectation if $\Phi(\sup X_\alpha)=\sup \Phi(X_\alpha)$ where X_α is a set of uniformly bounded selfadjoint operators of M. Let $\{\Phi_\alpha\}$ be a set of expectations of M on N. $\{\Phi_\alpha\}$ is said to form a complete set if $\Phi_\alpha(X)=0$ for all α and X positive in M, imply $X=0$. A state ρ of M will be a positive linear functional on M of norm one. As above one can define normal and faithful states and complete families of states. ρ is said to diagonalize N if $\rho(AX)=\rho(XA)$ for all X in M and all A in N. Let Z be the center of N.

2. Results.

THEOREM 1. *Suppose there exists a set of states* $\{\rho_\alpha\}$ *of* M *satisfying*:
(1) *Each* ρ_α *diagonalizes* N,
(2) *Each* ρ_α *restricted to* N *is normal*,
(3) *The set of* ρ_α *is complete. Then there exists a complete set of expectations from* M *to* N *and* N *is a finite von Neumann algebra. If all* ρ_α *are normal in* M *all these expectations are normal. Conversely if* N *is finite and if there exists a complete set of expectations of* M *on* N, *then there exists a set of states of* M *satisfying* (1), (2), (3).

THEOREM 2. *Let* M *be a von Neumann subalgebra of* $\mathcal{L}(h)$ *then if there exists a complete set of expectations of* $\mathcal{L}(h)$ *on* M, M *is atomic. Conversely if* M *is an atomic von Neumann subalgebra of* $\mathcal{L}(h)$, *then there exists a complete set of normal expectations of* $\mathcal{L}(h)$ *on* M.

COROLLARY. *There exists a complete set of expectations of* $\mathcal{L}(h)$ *on* M *if there exists a complete set of expectations of* $\mathcal{L}(h)$ *on the commutant of* M.

N^c will now denote the relative commutant of N in M i.e., $N^c=N'\cap M$ where N' are all operators of $\mathcal{L}(h)$ which commute with N.

Let $\mathcal{U}(\mathcal{A})$ denote the unitary group of a von Neumann algebra $\mathcal{A}$. Now let G be a subgroup of $\mathcal{U}(M)$. By a Schwartz map relative to

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(G, M) is meant a linear map P of M into itself such that

(1) $P(X) = UP(X)U^{-1}$ for all U in G and all X in M.

(2) $P(X)$ belongs to the weak closure of the convex hull generated by elements of the form UXU^{-1} as U ranges over G.

$S(G, M)$ will designate all Schwartz maps relative to (G, M) which are G-stable i.e., $P(X) = P(VXV^{-1})$ for all V in G. $S(G, M)$ will be called sufficient if for any positive X in M, $P(X) = 0$ for all P in $S(G, m)$ implies $X = 0$.

THEOREM 3. *Suppose M has a faithful, normal, semi-finite trace; and suppose $S(G, M)$ is sufficient. Then there exists a faithful, normal $\mathfrak{U}(N^{cc})$-stable expectation of M on N^c.*

COROLLARY. *With the above hypothesis, if M is of type* I *so is N^c.*

Now the following uniqueness theorem can be stated:

THEOREM 4. *Assume the following conditions to hold:*

(1) $N' \cap M \subset N$,

(2) *N is finite,*

(3) *M is semifinite.*

Then there exists at most one normal expectation of M in N.

In what follows $\otimes$ will mean tensor product. An ampliation of M in $\mathfrak{L}(h \otimes k)$ where h is the Hilbert space on which M acts and k is any Hilbert space is defined to be the map which sends X into $X \otimes I_k$ where X is in M and I_k is the identity map on K. Let T_1 and T_2 be two bounded maps acting on two Hilbert spaces h_1 and h_2. T_1 and T_2 are called spatially isomorphic if there exists an isometry V of h_1 on h_2 such that $T_1 = V^* T_2 V$.

THEOREM 5. *There exists an ampliation of M in $\mathfrak{L}(h \otimes k)$ such that if N is any von Neumann subalgebra of M which is the range of a faithful, normal expectation Φ, then there exists an isometry V in the commutant of $N \otimes I_k$ such that $\Phi \otimes I_k$ is spatially isomorphic via V to the identity map, i.e. $\Phi(X \otimes I_k) = V(X \otimes I_k)V^*$, $VV^* = I$. On putting $V^*V = P$ then P is in the commutant of $N \otimes I_k$ and*

$$(\Phi \otimes I_k)(X \otimes I_k)P = P(X \otimes I_k)P;$$

for all positive $X \otimes I_k$, $(X \otimes I_k)P = 0$ implies $X \otimes I_k = 0$.

This theorem essentially states that by an ampliation a faithful normal expectation carries over to a map spatially isomorphic with the identity map. Moreover the ampliation is independent of the particular expectation.

3. Remarks. (a) The result of Theorem 5 is an improvement of a result contained in [2]. In [2] the case when M was finite and countably decomposable was considered.

(b) The proof of Theorem 3 is accomplished by use of the machinery developed in [1]. Central to the proof is a result by Sakai [3] which says that if M_p is uniformly separable, then M_p is finitely generated as an algebra. Finally a result by Tomiyama in [7] will be used.

(c) In [4] J. Schwartz defined maps that are not necessarily stable. By use of two results of [6] the corollary to Theorem 3 follows.

(d) In [9] Umegaki has studied uniqueness of expectations in case M is finite and countably decomposable.

(e) The proof of Theorem 5 depends on the Stinespring construction [5] and on a result by Tomiyama [8]. In [2] Nakamura, Takesaki, Umegaki study the case when M is finite, and the present result subsumes theirs.

The results of this paper are related to, but not subsumed under the results of [10]. The complete details of the above results will be published elsewhere.

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