ON SIMPLE GROUPS OF ORDER $5 \cdot 3^a \cdot 2^b$

BY RICHARD BRAUER[1]

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The following theorem can be proved.

THEOREM. *If G is a simple group of an order g of the form $g = 5 \cdot 3^a \cdot 2^b$, $g \neq 5$, then G is isomorphic to one of the alternating groups A_5, A_6, or to the group $O_5(3)$ of order* 25,920.

One may conjecture that there exist only finitely many nonisomorphic noncyclic groups whose order g is divisible by exactly three distinct primes $p<q<r$. J. G. Thompson [6] has shown that then $p=2$, $q=3$ while r is 5, 7, 13, or 17. It is not unlikely that if one of the exponents a,b,c is 1, the methods applied here can be used to find all simple groups of the orders in question. No example is known in which all three exponents a,b,c are larger than 1.

Since the proof of the theorem is long, we do not intend to publish it. A complete account has been prepared in mimeographed form.[2] We shall give a brief outline.

1. We start with two propositions of slightly more general interest.

PROPOSITION 1. *Let G be a simple group of an order $g = p^a q^b r^c$ where p,q,r are distinct primes. Assume that the Sylow-subgroup R of G of order r^c is cyclic. Then R is self-centralizing in G; $C(R) = R$.*

PROOF. If this was false, we may assume that $C(R)$ contains an element π of order p, (interchanging p and q, if necessary). Then, for $R = \langle \rho \rangle$,

$$\sum \chi_j(\pi\rho)\chi_j(1) = 0$$

where χ_j ranges over the irreducible characters of G in the principal p-block $B_0(p)$. It follows that there exists a nonprincipal character $\chi_j \in B_0(p)$ such that

$$(1) \qquad \chi_j(1) \not\equiv 0 \pmod{q}, \qquad \chi_j(\pi\rho) \neq 0.$$

If here χ_j belongs to the r-block $B(r)$, the second condition (1) implies that ρ belongs to a defect group D of $B(r)$, cf. [2]. Thus, $D = R$. It

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[2] This report can be obtained on request from the Department of Mathematics, Harvard University, Cambridge, Massachusetts 02138.

follows that $\chi_j(1)\not\equiv 0 \pmod r$, cf. [3] or [5]. Hence $\chi_j(1)$ is a power of p. This is impossible for $\chi_j\in B_0(p)$, cf. [4], and the proposition is proved.

Let θ be a class function defined on a finite group. Then θ is a linear combination

$$\theta = \sum_i c_i\chi_i \tag{2}$$

of the irreducible characters χ_i of G with complex coefficients. If B is a p-block of G for some prime p, we shall denote by θ_B the expression obtained if we let χ_i range in (2) only over the characters $\chi_i\in B$. Hence

$$\theta = \sum \theta_B \tag{3}$$

where B ranges over all p-blocks of G.

Proposition 2. *Let G be a finite group of an order $g=p^n g_1 g_2$, (p a prime, g_1 and g_2 positive integers). Let θ and η be class functions on G with $\theta(1)\neq 0$, $\eta(1)\neq 0$, such that θ vanishes for all elements of G of an order divisible by some prime factor of g_1 and that η vanishes for all elements of G of an order divisible by some prime factor of g_2. Then there exists a p-block B for which $\theta_B\eta_B\neq 0$.*

Proof. Let ρ range over the p-regular elements of G. It follows from the assumptions that $\theta(\rho)\eta(\rho)=0$ for $\rho\neq 1$. Hence

$$\sum_\rho \theta(\rho)\eta(\rho) = \theta(1)\eta(1) \neq 0. \tag{4}$$

Let B and $B'\neq B$ be two p-blocks of G. If we express θ and η by the p-modular characters of G, the orthogonality relations show that

$$\sum_\rho \theta_B(\rho)\theta_{B'}(\rho) = 0. \tag{5}$$

Our result is obtained by combining (3), the analogous relation for η, (4), and (5).

2. Assume now that G satisfies the hypotheses of the theorem. It follows from Proposition 1 that if $R=\langle\rho\rangle$ is a subgroup of order 5 of G, then R is self-centralizing. This implies that the principal 5-block $B_0(5)$ is the only 5-block of G of positive defect. Set

$$\psi = \sum \chi_i(\rho)\chi_i$$

with χ_i ranging over $B_0(5)$. Then ψ vanishes for all 5-regular elements of G. On account of Proposition 1, ψ then vanishes for all 2-singular elements of G. This implies that if $B(2)$ is a 2-block of G, $\psi_{B(2)}$ vanishes for all 2-singular elements. Likewise, if $B(3)$ is a 3-block, $\psi_{B(3)}$ vanishes for all 3-singular elements.

A great deal of information is available concerning the characters $\chi_i \in B_0(5)$, cf. [1] or [5]. It follows at once that $B_0(5)$ contains an irreducible character χ_n of degree $3^\alpha > 1$ and an irreducible character χ_h of degree $2^\beta > 1$. Here χ_n belongs to a 3-block $B^*(3)$ different from the principal 3-block and χ_h belongs to a 2-block $B^*(2)$ different from the principal 2-block [4].

The normalizer $N(R)$ of R in G has either order 10 or 20. In the former case, we have $3^\alpha - 2^\beta = \pm 1$. Then $\alpha \leqq 2$. For $B(3) = B^*(3)$, $\psi_{B^*(3)} = \pm \chi_n$. Hence χ_n vanishes for all 3-singular elements. This implies $\alpha = a$. Likewise, $\beta = b$. It follows that $g = 60$ or 360. Then $G \simeq A_5$ or $G \simeq A_6$ respectively.

The discussion of the case $|N(R)| = 20$ is more difficult. Here, $B_0(5)$ consists of five irreducible characters χ_i, and $\chi_i(\rho) = \pm 1$, $\chi_i(1) \equiv \chi_i(\rho) \pmod 5$. There are several ways in which Proposition 2 can be applied. For instance, let $B(2)$ be a 2-block which meets $B_0(5)$ and let $B(3)$ be a 3-block such that

$$B_0(5) \cap B(2) \cap B(3) = \emptyset. \tag{6}$$

We claim that $B(3) \subseteq B_0(5)$. Indeed, if χ_i was an irreducible character in $B(3)$ and not in $B_0(5)$, then $\theta = \chi_i$ vanishes for all 5-singular elements of G while $\eta = \psi_{B(2)}$ vanishes for all 2-singular elements. Now Proposition 2 with $p = 3$ yields a contradiction with (6). Similarly, we see that under the same assumptions, all irreducible characters in $B(3)$ have the same degree.

If we take $B(2) = B^*(2)$, $B(3) = B^*(3)$ and if (6) holds, then $B^*(3)$ consists entirely of characters of degree 3^α. There are then necessarily $3^{a-\alpha}$ members of $B^*(3)$. If the degree 3^α occurs only once, $\alpha = a$. Analogous results hold for $B^*(2)$.

Finally, it is easy to obtain inequalities for the degrees of the irreducible characters in $B_0(5)$. For instance, it can be shown that there exists an irreducible character $\chi_\lambda \in B_0(5)$ such that the five degrees in suitable order are at most equal to

$$\chi_\lambda(1)^0 = 1, \quad \chi_\lambda(1), \quad \chi_\lambda(1)^2, \quad \chi_\lambda(1)^3, \quad \chi_\lambda(1)^4$$

respectively. Combining our results with arguments from elementary number theory, we can show that the five degrees in $B_0(5)$ are

$$1, \ 6, \ 24, \ 64, \ 81.$$

It follows that $\alpha = a = 4$, $\beta = b = 6$ and that

$$g = 5\cdot 81\cdot 64 = 25{,}920.$$

3. It still remains to identify the group G. If σ,τ,ξ are elements of G, if χ_i ranges over all irreducible characters of G, it is well known that

$$(7) \qquad a(\sigma, \tau, \xi) = g\,|\,C(\sigma)\,|^{-1}\cdot\,|\,C(\tau)\,|^{-1}\sum \chi_i(\sigma)\chi_i(\tau)\chi_i(\xi)/\chi_i(1)$$

is a nonnegative rational integer. Indeed, $a(\sigma,\tau,\xi)$ is equal to the number of representations of ξ as a product st of a conjugate s of σ and a conjugate t of τ. If we choose ξ as an element ρ of order 5, then $\chi_i(\rho) = 0$ for $\chi_i \notin B_0(5)$ while $\chi_i(\rho)$ is known for $\chi_i \in B_0(5)$. Using this and other known properties, we can discuss the values of the characters $\chi_i \in B_0(5)$ for other elements of G. In particular, we can show that there exist elements μ of order 4 and ν of order 3 with

$$|\,C(\mu)\,| = 8, \qquad |\,C(\nu)\,| = 9.$$

It is then easy to see that for some irreducible character $\chi_k \neq 1$ of G both $\chi_k(\mu)$ and $\chi_k(\nu)$ are units. This implies that χ_k has degree 5. A final discussion shows that χ_k takes rational values for 3-regular elements of G.

The irreducible representation X with the character χ_k gives rise to a 3-modular representation Y of G of degree 5; the character ϕ of Y is the restriction of χ_k to the set of 3-regular elements of G. Since ϕ takes only rational values, Y can be written in the Galois field with 3 elements and Y possesses a nontrivial bilinear invariant. It is then easy to see that Y has a nontrivial quadratic invariant and it follows that $G \simeq O_5(3)$.

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Harvard University