TOEPLITZ AND WIENER-HOPF OPERATORS IN $H^\infty + C$

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1. Let L^p denote the Lebesgue space for the normalized measure $(1/2\pi)d\theta$ defined on the unit circle $\boldsymbol{T} = \{e^{i\theta} \mid 0 \leqq \theta \leqq 2\pi\}$, let H^p denote the corresponding Hardy space of functions in L^p which have zero negative Fourier coefficients and let P be the projection of L^2 onto H^2. For ϕ in L^∞ we define the Toeplitz operator T_ϕ on H^2 by $T_\phi f = P(\phi f)$ for f in H^2. It is clear that T_ϕ is bounded and this class of operators has been much studied.

Much of the interest in Toeplitz operators has been directed to the determination of their spectra. For ϕ in H^∞ it was shown by Wintner in [12] that the spectrum $\sigma(T_\phi)$ of T_ϕ is the closure of the range of the analytic extension of ϕ to the unit disk $\boldsymbol{D}$. For ϕ in the space C of continuous functions on $\boldsymbol{T}$ it was shown by Devinatz [4] (see [10] for earlier results) that $\sigma(T_\phi)$ consists of the range of ϕ on $\boldsymbol{T}$ along with those λ for which the index of λ with respect to the curve determined by ϕ is different from zero. In this note we describe $\sigma(T_\phi)$ for ϕ in the linear span of H^∞ and C. (This manifold is actually a closed subalgebra of L^∞.) We show that such a T_ϕ is invertible if its harmonic extension $\hat{\phi}$ to $\boldsymbol{D}$ is bounded away from zero on a neighborhood of $\boldsymbol{T}$ and the index of the curve $\hat{\phi}(Re^{i\theta})$ is zero for R sufficiently large. Our technique can be viewed as an extension of that used in [6] to determine $\sigma(T_\phi)$ for ϕ in C.

In §3 we indicate how to extend our results to determine the index of a certain class of vector-valued Toeplitz operators (or systems of Toeplitz operators). In this we generalize certain of the results of Gohberg and Kreĭn in [7]. We conclude by describing how our results can be applied to the study of Wiener-Hopf operators both in the scalar case and the vector-valued case using the isomorphism exhibited in [5].

We only outline our proofs and complete details will appear elsewhere.

2. We begin by recalling some facts about Fredholm operators. Let $\mathcal{L}$ denote the algebra of bounded operators on H^2, $\mathcal{K}$ the uniformly closed two-sided ideal of compact operators in $\mathcal{L}$, and π the homomorphism of $\mathcal{L}$ onto $\mathcal{L}/\mathcal{K}$. An operator A in $\mathcal{L}$ is said to be a Fredholm operator if A has a closed range and both a finite dimen-

sional kernel and cokernel. It is known [1] that this is equivalent to $\pi(A)$ being an invertible element of $\mathfrak{L}/\mathfrak{K}$. If A is a Fredholm operator, then the analytical index $i_a(A)$ is defined $i_a(A) = \dim [\ker A] - \dim [\ker A^*]$, where ker ( ) denotes the kernel.

One reason the notion of index is important for determining the invertibility of Toeplitz operators is the following fact proved by Coburn [2].

LEMMA 1. *For ϕ in L^∞ either* $\ker T_\phi = (0)$ *or* $\ker T_\phi^* = (0)$.

Thus, if T_ϕ is known to be a Fredholm operator, then T_ϕ is invertible if and only if $i_a(T_\phi) = 0$. We shall show for ϕ in $H^\infty + C$ that T_ϕ is a Fredholm operator if and only if ϕ is an invertible element of the algebra $H^\infty + C$. That $H^\infty + C$ is an algebra is a result due to Sarason [9] which we state as a lemma.

LEMMA 2. *The linear span of H^∞ and C is a closed subalgebra of L^∞. Moreover, the maximal ideal space of $H^\infty + C$ is the maximal ideal space of H^∞ with the unit disk removed.*

Sarason shows in [9] that the linear span of H^∞ and C is closed. The observation that the "closure" of $H^\infty + C$ coincides with the closed subalgebra of L^∞ generated by H^∞ and $\bar{z}$ allows us to conclude that $H^\infty + C$ is an algebra and to identify its maximal ideal space. The latter is a special case of the following proposition which is itself of interest. Let X be a compact Hausdorff space and $\mathfrak{A}$ be a uniformly closed subalgebra of the space $C(X)$ of continuous complex functions on X which separates points and contains the constants. Let ϕ be a function on X having modulus one in $\mathfrak{A}$ and let $\mathfrak{A}(\bar{\phi})$ denote the closed subalgebra of $C(X)$ generated by $\mathfrak{A}$ and $\bar{\phi}$. Then the maximal ideal space for $\mathfrak{A}(\bar{\phi})$ is obtained from that of $\mathfrak{A}$ by deleting the open set on which the Gelfand transform of ϕ has modulus less than one.

Now let $\mathfrak{G}$ denote the uniformly closed subalgebra of $\mathfrak{L}$ generated by the operators T_ϕ with ϕ in $H^\infty + C$. Note that $\mathfrak{G}$ is not a C^*-algebra.

LEMMA 3. *The algebra $\mathfrak{G}$ contains $\mathfrak{K}$ as a two-sided ideal and $\mathfrak{G}/\mathfrak{K}$ is isometrically isomorphic to $H^\infty + C$.*

PROOF. Since $\mathfrak{G}$ contains the C^*-algebra generated by the unilateral shift of multiplicity one, it follows from [3] that $\mathfrak{G}$ contains $\mathfrak{K}$ and $\mathfrak{K}$ is an ideal in any algebra of $\mathfrak{L}$ containing it. If p and q are trigonometric polynomials and ψ and ζ are functions in H^∞, then a straightforward computation shows that the commutator of $T_{\psi+p}$ and $T_{\zeta+q}$ is compact. Thus the *linear* span of the operators of the form $T_\phi + K$, where ϕ is in $H^\infty + C$ and K is in $\mathfrak{K}$, is an algebra. That it is in fact a closed algebra follows from the inequality $\|T_\phi+K\|\geqq\|T_\phi\|$ proved in [2]. Therefore, $\mathcal{G}/\mathcal{K}$ is commutative and the mapping $T_\phi+K\leftrightarrow\phi$ is an isometrical isomorphism of $\mathcal{G}/\mathcal{K}$ onto $H^\infty+C$.

COROLLARY. *If ϕ is in $H^\infty+C$, then $T_\phi-\lambda$ is a Fredholm operator if and only if $\phi-\lambda$ is an invertible element of $H^\infty+C$.*

PROOF. If $\phi-\lambda$ is an invertible element of $H^\infty+C$, then it follows from the preceding lemma that $T_\phi-\lambda$ is a Fredholm operator. Conversely, if $T_\phi-\lambda$ is a Fredholm operator, then $\pi(T_\phi-\lambda)$ is an invertible element of $\mathcal{L}/\mathcal{K}$ and we must show that its inverse is in $\mathcal{G}/\mathcal{K}$. This can be shown for a ϕ in H^∞ and the problem for an arbitrary ϕ in $H^\infty+C$ is solved by approximating ϕ by a function of the form $z^{-n}\psi$ where ψ is in H^∞.

The preceding result determines when $T_\phi-\lambda$ is a Fredholm operator. This combined with Lemma 1 will enable us to determine $\sigma(T_\phi)$ when we have some effective method of determining the index of $T_\phi-\lambda$. If ϕ is continuous, the index of $T_\phi-\lambda$ is equal to the negative of the topological index $i_t(\phi, \lambda)$ of the curve determined by ϕ with respect to λ (cf. [6]). In the case at hand we use the index of the curves $\hat{\phi}(re^{i\theta})$ where $\hat{\phi}$ is the harmonic extension of ϕ to the interior of $\boldsymbol{D}$. To this end we need to relate the invertibility of ϕ in $H^\infty+C$ to the function $\hat{\phi}$ on $\boldsymbol{D}$.

LEMMA 4. *A necessary and sufficient condition that ϕ in $H^\infty+C$ be invertible is that ϕ^{-1} be in L^∞ and for each $\epsilon>0$, there exists $\delta>0$ so that $|\hat{\phi}(re^{i\theta})|\geqq 1/\|\phi^{-1}\|_\infty-\epsilon$ for $1>r\geqq 1-\delta$.*

PROOF. We again approximate ϕ by a function of the form $z^{-n}\psi$ with ψ in H^∞ and analyze the inner and outer factors of ψ.

LEMMA 5. *If $T_\phi-\lambda$ is a Fredholm operator, then $i_a(T_\phi-\lambda) = -\lim_{R\to 1^-} i_t(\hat{\phi}(Re^{i\theta}), \lambda)$.*

PROOF. From the Corollary and Lemma 4 it follows that for $T_\phi-\lambda$ a Fredholm operator there exists $0<R<1$ so that $\hat{\phi}(re^{i\theta})\neq\lambda$ for $1>r\geqq R$. Since $\hat{\phi}$ is continuous on D we then have that $i_t(\hat{\phi}(re^{i\theta}), \lambda)$ is constant for $r\geqq R$ so that the limit exists. The proof is now accomplished with the same technique used in the preceding proof.

We can now determine the spectrum of T_ϕ for ϕ in $H^\infty+C$.

THEOREM. *For ϕ in $H^\infty+C$ we have T_ϕ is invertible if and only if*

$$\lim_{R\to 1^-} \inf_{0\leq\theta<2\pi;\, R\leqq r<1} |\hat{\phi}(re^{i\theta})| = \eta > 0$$

and

$$\lim_{R\to 1^-} i_t(\hat{\phi}(Re^{i\theta}), 0) = 0.$$

COROLLARY. *For ϕ in $H^\infty + C$ we have*

$$\sigma(T_\phi) = \{\lambda \mid \lim_{R\to 1^-} \inf_{0\leqq\theta<2\pi; R\leqq r<1} |\hat{\phi}(re^{i\theta}) - \lambda| = 0\}$$
$$\cup \{\lambda \mid \lim_{R\to 1^-} i_t(\hat{\phi}(Re^{i\theta}), \lambda) \neq 0\}.$$

We make several comments before continuing to the vector valued case. Firstly, although the statement of the Theorem makes sense for an arbitrary ϕ in L^∞ the Theorem is not valid in this generality. Secondly, $\lim_{r\to 1^-} i_t(\hat{\phi}(re^{i\theta}), \lambda) = 0$ does not imply that T_ϕ is invertible even for ϕ in H^∞. Thirdly, Widom has shown that $\sigma(T_\phi)$ is connected for ϕ in L^∞ (cf. [11]). We remark that it follows from the Corollary to Lemma 3, Lemma 1 and the fact that the maximal ideal space of $H^\infty + C$ is connected [8] that $\sigma(T_\phi)$ is connected for ϕ in $H^\infty + C$. Lastly, using the identification of Wiener-Hopf operators with Toeplitz operators (cf. [5]), our Theorem can be used to determine the invertibility of a certain class of Wiener-Hopf operators.

We now describe the extension of our results to the vector valued case. Let $\mathfrak{E}$ be a finite dimensional Hilbert space and $\mathcal{L}(\mathcal{E})$ the ring of bounded operators on $\mathfrak{E}$. Let $L^2_{\mathfrak{E}}$ denote the Hilbert space of measurable $\mathfrak{E}$-valued functions on T having square integrable norm, $H^2_{\mathfrak{E}}$ the corresponding Hardy space of functions in $L^2_{\mathfrak{E}}$ which have zero negative Fourier coefficients, and P the projection of $L^2_{\mathfrak{E}}$ onto $H^2_{\mathfrak{E}}$. Further, let $L^\infty_{\mathcal{L}(\mathcal{E})}$ denote the ring of bounded measurable $\mathcal{L}(\mathcal{E})$-valued functions on T and $H^\infty_{\mathfrak{E}}$ the Hardy space of functions in $L^\infty_{\mathcal{L}(\mathcal{E})}$ with zero negative Fourier coefficients. For Φ in $L^\infty_{\mathcal{L}(\mathcal{E})}$ we define the Toeplitz operator T_Φ on $H^2_{\mathfrak{E}}$ by $T_\Phi f = P(\Phi f)$, where Φf denotes the pointwise product. Finally, let $C_{\mathcal{L}(\mathcal{E})}$ denote the space of continuous $\mathcal{L}(\mathcal{E})$-valued functions on T.

THEOREM. *If Φ is in $H^\infty_{\mathcal{L}\mathcal{E}} + C_{\mathcal{L}\mathcal{E}}$ then T_Φ is a Fredholm operator if and only if*

$$\lim_{R\to 1^-} \inf_{0\leqq\theta<2\pi; R\leqq r<1} |(\det \Phi)^\wedge(Re^{i\theta})| = \eta > 0$$

and this case

$$i_a(T_\Phi) = -\lim_{R\to 1^-} i_t((\det \Phi)^\wedge(Re^{i\theta})).$$

PROOF. We briefly describe the changes necessary in the proof given for the scalar case. We define $\mathcal{G}$ as the closed subalgebra of $\mathcal{L}(H^2_{\mathfrak{E}})$

generated by the T_Φ for Φ in $\boldsymbol{H}^\infty_{\mathfrak{L}(\mathcal{E})}+C_{\mathfrak{L}(\mathcal{E})}$ and show $\mathcal{G}$ contains as an ideal, the ring $\mathcal{K}$ of compact operators on $\boldsymbol{H}^2_{\mathfrak{E}}$. Further, we show that $\mathcal{G}/\mathcal{K}$ is isometrically isomorphic to the closed subalgebra $\boldsymbol{H}^\infty_{\mathfrak{L}(\mathcal{E})}+C_{\mathfrak{L}(\mathcal{E})}$ of $\boldsymbol{L}^\infty_{\mathfrak{L}(\mathcal{E})}$. Again we show that $\pi(T_\Phi)$ is invertible in $\mathcal{G}/\mathcal{K}$ if and only if it is invertible in $\mathfrak{L}/\mathcal{K}$. Thus we find that T_Φ is a Fredholm operator if and only if $\det(\Phi)$ is an invertible element of $H^\infty+C$. Lastly, the index of T_Φ is computed using the fact that the determinant defines an isomorphism from the first homotopy group of the general linear group for $\mathfrak{E}$ onto the first homotopy group of the space of nonzero complex numbers.

Again using the isomorphism exhibited in [5] we can identify the operator T_Φ with a matrix valued Wiener-Hopf operator. In this context Gohberg and Kreĭn proved the preceding theorem for Φ in a certain subset of $C_{\mathfrak{L}(\mathcal{E})}$.

Complete details will appear elsewhere along with extensions of the preceding results to the case of an infinite dimensional space $\mathfrak{E}$ as well as to Toeplitz operators defined on certain other Banach spaces.

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