ON THE DERIVATIVE OF A SEMIGROUP[1]

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Let I be a countably infinite set, and $P=\{P(t, i, j)\}$ a standard semigroup on I: that is, $P(t)$ is a stochastic matrix, $P(t+s)=P(t)P(s)$, and

$$\lim_{t\to 0} P(t, i, i) = P(0, i, i) = 1 \quad \text{for all } i \in I.$$

As is well known, $Q=P'(0)$ exists, although $q(i)=Q(i, i)$ may be infinite for some or all i. When $q(i)<\infty$, the numbers $q(i)$ and $Q(i, j)/q(i)$ have interesting known probabilistic interpretations, although the meaning of $Q(i, j)$ itself is a little obscure. The object of this note is to "explain" $Q(i, j)$ in a way which does not depend on $q(i)$, finite or infinite.

To state the explanation, give I the discrete topology, and let $I\cup\{\phi\}$ be the one-point compactification. On a suitable probability triple, say $(\Omega, \mathfrak{F}, P_k)$, construct an $I\cup\{\phi\}$-valued process X, which is Markov with stationary transitions P, starts from $k\in I$, and has smooth sample functions.

More formally, for $0=t_0<t_1<\cdots<t_n$ and $i_0=k$ and $i_1, \cdots, i_n$ in I,

$$P_k\{X(t_m) = i_m \text{ for } m = 0, \cdots, n\} = \prod_{m=0}^{n-1} P(t_{m+1} - t_m, i_m, i_{m+1}).$$

Moreover, for each ω and all $t>0$, as rational r increases to t, the (generalized) sequence $X(r, \omega)$ has at most one limiting value in I. (This does not exclude the possibility of having ϕ as a limiting value or even converging to ϕ.) Finally, for each ω and all $t\geqq 0$, as rational r decreases to t, there are only two possibilities: either $X(t, \omega)=\phi$ and $X(r, \omega)$ tends to ϕ; or $X(t, \omega)\in I$ and $X(r, \omega)$ has precisely one limiting value in I, namely $X(t, \omega)$.

As is known, such a construction is always possible.

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Fix $i \neq j$ in I. Informally, let β be the time spent in i while waiting for the first pseudojump from i to j. Formally, let $\alpha(\omega)$ be the least t, if any, such that $X(t, \omega) = j$, while as rational r increases to t, the generalized sequence $X(r, \omega)$ has i for a limiting value. Let $\alpha(\omega) = \infty$ if no such t exists. Let

$$\beta(\omega) = \text{Lebesgue}\{t: 0 \leqq t \leqq \alpha(\omega) \text{ and } X(t, \omega) = i\}.$$

Verify that β is well defined and measurable; of course, $\beta = \infty$ is a possibility.

THEOREM 1. *If* $\int_0^\infty P(t, i, i)dt = \infty$, *then* β *is exponentially distributed with parameter* $Q(i, j)$, *namely*

$$P_i\{\beta \geqq t\} = e^{-Q(i,j)t}.$$

To handle the transient case, introduce the following definitions. Let [hit i] be the set of ω such that $X(t, \omega) = i$ for some $t \geqq 0$. Let

$$\gamma(\omega) = \text{Lebesgue } \{t: 0 \leqq t < \infty \quad \text{and} \quad X(t, \omega) = i\}$$

and

$$p^{-1} = \int_0^\infty P(t, i, i)dt.$$

THEOREM 2. *If* $\int_0^\infty P(t, i, i)dt < \infty$, *then* γ *is exponentially distributed with parameter* p, *relative to* P_i, *that is*,

$$P_i\{\gamma \geqq t\} = e^{-pt}.$$

Moreover, β *is exponentially distributed with parameter*

$$\lambda = p + Q(i, j)P_j(\text{hit } i),$$

relative to P_i. *In other words*, $\beta / P_i(\alpha < \infty)$ *is exponentially distributed with parameter* $Q(i, j)$, *relative to* P_i.

THEOREM 3. $Q(i, j) = 0$ *iff* $P_i(\alpha = \infty) = 1$.

By arguing more vigorously, it is possible to prove the following result. For simplicity, suppose $\int_0^\infty P(t, i, i)dt = \infty$. Let β_{jn} be the time spent in i until the nth pseudojump from i to j.

THEOREM 4. *The process* $\{\beta_{jn}: n = 1, 2, \cdots\}$ *is a Poisson process of points with rate* $Q(i, j)$. *As* j *varies, these processes are independent.*

Let K be a finite subset of I, and suppose $\sigma = \sum_{k \in k} Q(i, k) > 0$. Let θ be the time spent in i until the first pseudojump from i to some $k \in K$.

COROLLARY. *The random variable θ is exponential with parameter σ. The process X pseudojumps from i to $j \in K$ before pseudojumping to any other $k \in K$ with probability $Q(i, j)/\sigma$.*

Here is a brief outline of the proof for Theorem 4. I expect to publish detailed proofs elsewhere. Let $\{\beta_n: n=1, 2, \cdots\}$ be a Poisson process of points with rate q, that is, $\beta_1, \beta_2-\beta_1, \beta_3-\beta_2, \cdots$ are independent and exponentially distributed, with parameter q. Let F be a finite set, and p a probability on F. Let $Z_1, Z_2, \cdots$ be independent random variables with common distribution p. Suppose the process Z is independent of the process β. For $j \in F$, define a point process $\{\beta_{jn}: n=1, 2, \cdots\}$ as follows: β_{jn} is the nth β_m for which Z_m is j. That is, let $\phi(j, n)$ be the least m for which exactly n of $Z_1, \cdots, Z_m$ are equal to j, and let $\beta_{jn}=\beta_{\phi(j,n)}$. If $p(j)>0$, suppose that $Z_n=j$ for infinitely many n, so β_{jn} is well defined. If $p(j)=0$, suppose $Z_n=j$ for no n, and let the process β_j have no points.

PRINCIPLE. The process $\{\beta_{jn}: n=1, 2, \cdots\}$ is a Poisson process of points with rate $qp(j)$. As j varies, these processes are independent.

If I is finite, Theorem 4 follows from this principle. Let β be the point process whose inter-point times coincide with the successive holding times of X in i. Let F be the set $I-\{i\}$. Let $Z_n=j$ if X jumps to j on leaving the nth i-interval.

For infinite I, approximate X by the process X_J with finite state space $J \subset I$, obtained by deleting the times t at which $X(t) \notin J$. This method of approximation has been studied by Levy and Williams. I expect to discuss it elsewhere.

REFERENCE

1. David A. Freedman, *On the convergence of approximants to a Markov chain.* I, II, Proc. Nat. Acad. Sci. (1968) (to appear).

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