REPRESENTATION THEOREMS ON BANACH FUNCTION SPACES

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Let L_ρ be a Banach function space, i.e. a Banach space of (equivalence classes of) measurable point functions on a σ-finite measure space (Ω, Σ, μ), with ρ being a function norm possessing at least the weak Fatou property. The results obtained concern integral representations of bounded linear operators from a Banach space $\mathfrak{X}$ to L_ρ and from L_ρ (or a subspace) to $\mathfrak{X}$. These results in some cases complement and in other cases generalize work done in [1], [3], [5], [6], [7], [12], [13].

General notation and results on Banach function spaces can for the most part be found in the first parts of [11]; more detailed work is in [9]. A few further definitions are needed here. If $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces, let $B(\mathfrak{X}, \mathfrak{Y})$ be the space of bounded linear operators from $\mathfrak{X}$ to $\mathfrak{Y}$. Distinguish two subrings of Σ as $\Sigma_0 = \{E \in \Sigma: \rho(\chi_E) < \infty\}$ and $\Sigma_0' = \{E \in \Sigma: \rho'(\chi_E) < \infty\}$. A *partition* $\mathcal{E}$ is defined to be a finite disjoint collection of non-μ-null members of Σ_0 which are of finite measure. The *"averaged" step function* of a member f of L_ρ is defined as

$$f_{\mathcal{E}} = \sum_{\mathcal{E}} \left(\int_{E_i} |f| \, d\mu / \mu(E_i) \right) \chi_{E_i}.$$

A function norm ρ is said to have property (J) if, for each partition $\mathcal{E}$, $\rho(f_{\mathcal{E}}) \leq \rho(f)$. (This is very similar to the *levelling property* of [5].)

1. **The structure of the space $B(\mathfrak{X}, L_\rho)$.**

DEFINITION 1. We define a space of set functions: $\mathcal{V}_\rho = \{x^*(\cdot) \mid x^*(\cdot): \Sigma_0' \to \mathfrak{X}^*, x^*(\cdot)x$ is countably additive and μ-continuous for each $x \in \mathfrak{X}$, and $V_\rho(x^*(\cdot)) < \infty\}$ where

$$V_\rho(x^*(\cdot)) = \sup_{\|x\| \leq 1} \sup_{\mathcal{E}} \rho\left(\sum_{\mathcal{E}} \frac{x^*(E_i)x}{\mu(E_i)} \chi_{E_i} \right).$$

The representation of bounded linear operators from $\mathfrak{X}$ to L_ρ is made in terms of this space.

[1] The results announced here are contained in the author's doctoral dissertation written at Carnegie Institute of Technology under the guidance of Professor M. M. Rao.

THEOREM 1. *If ρ' possesses property* (J), *then there is an isomorphism between $B(\mathfrak{X}, L_\rho)$ and $\mathcal{V}_\rho$; moreover,*

$$\gamma\|T\| \leqq V_\rho(x^*(\cdot)) \leqq \gamma^{-1}\|T\|$$

for corresponding elements.

(The constant γ is fixed for each L_ρ space and has value $0<\gamma\leqq 1$ with $\gamma=1$ if and only if ρ has the strong Fatou property. It is the constant which appears in Theorems 1.1 and 1.2 of [11].)

The correspondence is obtained in one direction by defining $x^*(E)x=\int_E Tx(\omega)d\mu(\omega)$ for $E\in\Sigma_0'$. In the other direction, a type of Radon-Nikodym derivative is used.

2. **The structure of the space** $B(M_\rho, \mathfrak{X})$. Results for $B(L_\rho, \mathfrak{X})$ unfortunately seem not to be, in general, available by the present techniques. Results which will be presented in §3 have been obtained for the linear functionals ($\mathfrak{X}=$scalars). In the case of a general $\mathfrak{X}$, we have results for a closed subspace of L_ρ (which in some common cases is all of L_ρ).

DEFINITION 2. Let $M_\rho=\mathrm{cl}\{f\in L_\rho: f$ is bounded and has support in $\Sigma_0\}$.

Note that M_ρ is a closed subspace which is normal and a sublattice (in fact, a lattice ideal).

DEFINITION 3. We define a space of set functions: $\mathcal{W}_\rho'=\{x(\cdot)\,|\,x(\cdot): \Sigma_0\to\mathfrak{X}$, $x(\cdot)$ is finitely additive, vanishes on μ-null sets, and $W_\rho'(x(\cdot))<\infty\}$, where

$$W_\rho'(x(\cdot)) = \sup_{\|x^*\|<1}\ \sup_{\mathcal{E}} \rho'\left(\sum_{\mathcal{E}} \frac{x^*x(E_i)}{\mu(E_i)}\chi_{E_i}\right).$$

To represent elements of $B(M_\rho, \mathfrak{X})$ it will be desirable to integrate members of M_ρ against set functions in $\mathcal{W}_\rho'$. In order to do this, Bartle's treatment [2] of integration will be used.

DEFINITION 4. A measurable function f is integrable over Ω with respect to an $\mathfrak{X}$-valued finitely additive set function $x(\cdot)$ if there is a sequence $\{f_n\}$ of simple functions such that

(i) $f_n\to f$ in $x(\cdot)$ measure,

(ii) $\lambda_n(\cdot)$ are uniformly absolutely continuous, and

(iii) $\lambda_n(\cdot)$ are equicontinuous,

where $\lambda_n(E)=\int_E f_n dx$ for $E\in\Sigma$ and where (i), (ii), and (iii) are with respect to the semivariation of $x(\cdot)$.

The fact that if $f\in M_\rho$ and $x(\cdot)\in\mathcal{W}_\rho'$ then f is $x(\cdot)$ integrable leads to the representation theorem:

THEOREM 2. *If ρ' has* (J), *then $B(M_\rho, \mathfrak{X})$ and $\mathcal{W}_\rho'$ are isomorphic; moreover, $\|T\| \leqq W_\rho'(x(\cdot)) \leqq \gamma^{-1}\|T\|$ for corresponding elements.*

The correspondence is given by $Tf = \int f dx$ and $x(E) = T(\chi_E)$ for $E \in \Sigma_0$.

COROLLARY. *If every member of M_ρ has absolutely continuous norm, then each $x(\cdot) \in \mathcal{W}_\rho'$ is μ-continuous.*

3. **Representation of linear functionals.** We have obtained two characterizations of L^*. One assumes property (J), the other does not. Both results proceed by use of the quotient space $(L_\rho/M_\rho)^*$. Define $N_\rho = L_\rho/M_\rho$ and equip N_ρ with the usual factor norm and order (recalling that M_ρ is a lattice ideal). Denote the canonical map as $\lambda: L_\rho \to N_\rho$. Note that λ is continuous, interior, homomorphic (both linear and lattice), and has norm $\leqq 1$. In addition $\lambda^*: N_\rho \to M_\rho$ is an isometric isomorphic surjection. (Note that N_ρ is an AB lattice, even though it is not a Banach function space over the given measure space.) In L_ρ we define the convex, norm-determining and (in general) nonlinear subset $\tilde{L}_\rho = \{f: f = \bigvee_{i=1}^n f_i, f_i \geqq 0, \rho(f_i) \leqq 1, 1 \leqq n < \infty\}$. Since λ is interior, $\lambda(\tilde{L}_\rho)$ contains the nonnegative elements of the open unit ball of N_ρ. It is here that an assumption is needed:

CONDITION (I). $\lambda(\tilde{L}_\rho)$ lies in the closed unit ball of N_ρ. (With this assumption, N_ρ is an AL space in the sense of Kakutani.)

The characterization of N_ρ^* is in terms of certain additive set functions. Define ba(Ω, Σ, μ) to be the collection of bounded additive set functions on Σ which vanish on μ-null sets. Denote by ca(Ω, Σ, μ) the countably additive members of ba(Ω, Σ, μ) and by pfa(Ω, Σ, μ) the purely finitely additive members of ba(Ω, Σ, μ). We will need to integrate elements of N_ρ with respect to set functions in pfa(Ω, Σ, μ). The integration used is a variant of that found in [**14**] and [**15**].

DEFINITION 5. Let $0 \leqq \nu \in$ pfa(Ω, Σ, μ) and $0 \leqq f \in L_\rho$. Define $I_\nu(f) = \inf\{\sum_{i=1}^n \|\lambda(f\chi_{E_i})\| \nu(E_i): \{E_i\}$ disjoint finite partition of $\Omega\}$.

This has all the usually desired properties of an integral and is extended to all of L_ρ and pfa(Ω, Σ, μ) by linearity on the decomposition into their positive parts. Note that one could equally well write $I_\nu(\lambda(f))$ since I_ν is constant over cosets.

THEOREM 3. *Assuming that condition* (I) *holds, there is an isometric isomorphism which is also a lattice isomorphism between N_ρ^* and a closed subspace of* pfa(Ω, Σ, μ) *which shall be denoted as $\mathcal{P}_{\rho'}$. The isometry is $\|z^*\| = |\nu|(\Omega)$.*

The correspondence is given by: for $\nu \in \mathcal{P}_{\rho'}$, $z^*(\lambda(f)) = I_\nu(f)$, $f \in L_\rho$; for $0 \leqq z^* \in N_\rho^*$, $\nu(E) = \|z_E^*\|$, $E \in \Sigma$; and for general $z^* \in N_\rho^*$ one uses its decomposition into positive parts. (We denote $z_E^*(\lambda(f)) = z^*(\lambda(f)\chi_E)$.)

The space $\mathcal{P}_{\rho'}$ is determined as the range of the (bounded) projection obtained by composing the correspondence from pfa(Ω, Σ, μ) into N_ρ^* with that from N_ρ^* to pfa(Ω, Σ, μ). One may describe $\mathcal{P}_{\rho'}$ as those elements in pfa(Ω, Σ, μ) whose support lies inside the support of a function in $\tilde{L}_\rho$ which is not in M_ρ.

THEOREM 4. *The conjugate space L_ρ^* has a direct sum decomposition into two closed linear (lattice) subspaces which are seminormal, namely into $M_\rho^\perp$ and its lattice orthogonal complement $(M_\rho^\perp)^{\text{o.c.}}$ which is isometrically linearly and lattice isomorphic to M_ρ^*. Moreover, in the decomposition, $\|x^*\| = \|y^*\| + \|z^*\|$ where $x^* = y^* + z^*$ with $y^* \in M_\rho^\perp$ and $z^* \in (M_\rho^\perp)^{\text{o.c.}}$.*

Thus in order to represent L_ρ^* all that is needed is a representation of M_ρ^*. If ρ' has property (J), then by Theorem 2, M_ρ^* is isomorphic to the space of set functions $\mathcal{W}_{\rho'}^R = \{G \mid G\colon \Sigma_0 \to \text{reals}, G$ finitely additive on Σ_0, G vanishes on μ-null sets, and $\mathcal{W}_{\rho'}(G) = \sup_8 \rho'(\sum_8 (G(E_i)/\mu(E_i))\chi_{E_i}) < \infty\}$ under the correspondence $G(E) = x^*(\chi_E)$ for $E \in \Sigma_0$, and $x^*(f) = \int f dG$ for $f \in M_\rho$. For corresponding elements, one has $\|x^*\| \leqq W_{\rho'}(G) \leqq \gamma^{-1}\|x^*\|$. (It is also true that this correspondence is a lattice isomorphism.)

If we define $\mathcal{A}_{\rho'} = \mathcal{W}_{\rho'}^R \times \mathcal{P}_{\rho'}$ with norm $\|(G, \nu)\| = W_{\rho'}(G) + |\nu|(\Omega)$ and with partial order $(G, \nu) \geqq (0, 0)$ if $G \geqq 0$ and $\nu \geqq 0$, then $\mathcal{A}_{\rho'}$ is a Banach lattice and we have:

THEOREM 5. *If condition* (I) *holds and ρ' has property* (J), *then the space L_ρ^* is linear and lattice isomorphic to $\mathcal{A}_{\rho'}$. Moreover the correspondence is a topological equivalence.*

There is another characterization of M_ρ^* that is available without the assumption that ρ' has (J). However, this form is not as useful since the norm computation does not explicitly involve the associate space (although if ρ' has (J), this approach leads to a norm equivalent to the one given above).

THEOREM 6. *There is an isometric isomorphism between M_ρ^* and the Banach space $\mathcal{V} = \{\nu \mid \nu(\cdot)$ real valued, additive set function on Σ_0 which vanishes on μ-null sets, and $\|\nu\| = \sup[|\int f d\nu| : f$ simple and $\rho(f) \leqq 1] < \infty\}$. Moreover, the members of $\mathcal{V}$ are all countably additive if and only if every function in M_ρ is of absolutely continuous norm.*

If we define $\mathcal{B}_{\rho'}$ as $\mathcal{V} \times \mathcal{P}_{\rho'}$ with norm $\|(\nu, \psi)\| = \|\nu\| + |\psi|(\Omega)$ and partial order $(\nu, \psi) \geqq (0, 0)$ if $\nu \geqq 0$ and $\psi \geqq 0$, then $\mathcal{B}_{p'}$ is a Banach lattice and we have:

THEOREM 7. *Under the assumption of* (I) *alone, the space* L_p^* *is linearly and lattice isomorphic to* $\mathcal{B}_{p'}$. *Moreover the correspondence is an isometry.*

Details and proofs will appear in Trans. Amer. Math. Soc.

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[2] Notes XIV, XV, and XVI are by W. A. J. Luxemburg alone.