THE AXIOM OF DETERMINATENESS AND REDUCTION PRINCIPLES IN THE ANALYTICAL HIERARCHY

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Let R be the set of all sets of natural numbers. A collection $\mathcal{A}$ of subsets of R satisfies a *reduction principle* if, for every A and $B \in \mathcal{A}$, there are A' and $B' \in \mathcal{A}$ such that $A' \subseteq A$, $B' \subseteq B$, $A' \cup B' = A \cup B$, and $A' \cap B'$ is empty. For $n > 0$ let Π_n^1 and Σ_n^1 be, respectively the set of Π_n^1 subsets of R and the set of Σ_n^1 subsets of R. It is known that Π_1^1 and Σ_2^1 satisfy reduction principles and that for no n do both Π_n^1 and Σ_n^1 satisfy reduction principles. (For basic definitions and facts concerning the analytical hierarchy and the degrees of unsolvability, see [5].) Using the Axiom of Constructibility, Addison [1] shows that, for all $n \geqq 2$, Σ_n^1 satisfies a reduction principle. J. Silver has shown that Addison's result is consistent with the assertion that a measurable cardinal exists.

For each $n > 0$, let Γ_n^1 be Π_n^1 if n is odd and Σ_n^1 if n is even. For a statement of the Mycielski-Steinhaus Axiom of Determinateness (AD) and proofs of some of its consequences, see [4]. We assume AD and the Axiom of Dependent Choice (DC) and outline a proof that, for every n, Γ_n^1 (and hence $\mathbf{\Gamma}_n^1$) satisfies a reduction principle. This result has been proved independently by Moschovakis and Addison [2].

Since AD is false, a word is in order about the significance of our proof. In the notation of [4], AD says that, for every $P \subseteq 2^\omega$, $G_2(P)$ is determined. Although this contradicts the Axiom of Choice, it remains possible that a very large class of $G_2(P)$ are determined. For instance, it is possible that $G_2(P)$ is determined for every projective P, and this is enough to deduce our result. Indeed, to prove reduction for Γ_n^1 we need only assume that $G_2(P)$ is determined for every $\mathbf{\Delta}_{n-1}^1 P$. We need DC for $n \geqq 4$. While AD may well be consistent with DC, our justification for using DC is rather that we are assuming only a part of AD which we hope to be consistent with the Axiom of Choice.

Our tool in studying the analytical hierarchy is the Lemma below. Our first proof of reduction for Π_3^1 was based on a new proof by Blackwell [3] using infinite games—of reduction for Π_1^1. (The methods of [2] are closely related to those of Blackwell.) However, the Lemma provides a different proof which generalizes easily to all odd levels of the hierarchy. The Lemma is a consequence of AD and is an interesting proposition in its own right. Also, the problem of proving the Lemma consistent (say, assuming large cardinals of some kind) might be much easier than the consistency of full AD. (The Lemma is inconsistent with the Axiom of Choice.)

LEMMA. *Let $\mathcal{E}\subseteq\mathfrak{D}$, the set of all degrees of unsolvability. There is a degree $\boldsymbol{d}_0$ such that either $\boldsymbol{d}\geqq\boldsymbol{d}_0\rightarrow\boldsymbol{d}\in\mathcal{E}$ or $\boldsymbol{d}\geqq\boldsymbol{d}_0\rightarrow\boldsymbol{d}\in\mathfrak{D}-\mathcal{E}$.*

PROOF (Assuming AD). Let P be the set of all sequences $\in 2^\omega$ whose degree of unsolvability belongs to $\mathcal{E}$. We consider the game $G_2(P)$. Suppose for definiteness that I has a winning strategy. The strategy is essentially a number-theoretic function. Let $\boldsymbol{d}_0$ be its degree of unsolvability. We show that $\boldsymbol{d}\geqq\boldsymbol{d}_0\rightarrow\boldsymbol{d}\in\mathcal{E}$. Let $\boldsymbol{d}\geqq\boldsymbol{d}_0$ and let α be a sequence of degree $\boldsymbol{d}$. Suppose II plays α and I plays according to his strategy. The sequence produced has degree $\boldsymbol{d}$. Hence $\boldsymbol{d}\in\mathcal{E}$.

The Lemma yields a countably additive zero-one measure, which we call μ, on the degrees. $\mu(\mathcal{E})=1$ provided that every sufficiently large degree belongs to $\mathcal{E}$. (The *particular* measure μ is very important in the sequel. Many other measures on $\mathfrak{D}$ can be defined using AD.) To prove a reduction principle for each Γ_n^1, we simply use μ to continue past level 2 the familiar process of assigning ordinal numbers to Γ_n^1 sentences. This assignment of ordinals is enough not only to prove reduction principles but also to lift much of the theory of Π_1^1 and Σ_2^1 up to a theory of Γ_n^1. (See [2] for details.) The delicate theorems about Π_1^1, such as the Kondo-Addison Theorem, seem, however, to require more than just this assignment of ordinals.

For each $n>0$ and $m\geqq 0$, let W_n^m be the set of all $\langle e, \alpha_1, \cdots, \alpha_m\rangle$ such that e is the Gödel number of a Γ_n^1 formula A of m set variables and $A(\alpha_1, \cdots, \alpha_m)$ holds. We sometimes write $\langle e, \alpha_1, \cdots, \alpha_m\rangle$ as $\langle e, \boldsymbol{\alpha}\rangle$ or $A(\boldsymbol{\alpha})$. By induction on n, we assign ordinals $|e, \boldsymbol{\alpha}|^n$ to the members $\langle e, \boldsymbol{\alpha}\rangle$ of W_n^m. The assignment of ordinals will have two properties:

(1) The relations $|e_1, \boldsymbol{\alpha}_1|^n<|e_2, \boldsymbol{\alpha}_2|^n$ and $|e_1, \boldsymbol{\alpha}_1|^n\leqq|e_2, \boldsymbol{\alpha}_2|^n$ are Γ_n^1.

(2) For $\langle e_2, \boldsymbol{\alpha}_2\rangle\in W_n^m$, the complements of $|e_1, \boldsymbol{\alpha}_1|^n<|e_2, \boldsymbol{\alpha}_2|^n$ and $|e_1, \boldsymbol{\alpha}_1|^n\leqq|e_2, \boldsymbol{\alpha}_2|^n$ are Γ_n^1 uniformly in $\langle e_2, \boldsymbol{\alpha}_2\rangle$.

For $n=1$, the assignment of ordinals is a standard procedure. There is also a well-known method for using the assignment for Π_1^1 to get an assignment for Σ_2^1. This method is perfectly general and allows us to go from Π_n^1 to Σ_{n+1}^1: let $|(\exists\alpha)A(\alpha, \boldsymbol{\alpha})|^{n+1}=\inf\{|A(\alpha, \boldsymbol{\alpha})|^n: A(\alpha, \boldsymbol{\alpha})\}$. We omit the details.

Let $n\geqq 3$ be odd. Let $(\alpha)A(\alpha, \boldsymbol{\alpha})\in W_n^m$. For each $\alpha\in R$, let $\boldsymbol{d}(\alpha)$ be the degree of α. We assign to each degree of unsolvability $\boldsymbol{d}$ an ordinal $|A(\boldsymbol{d}, \boldsymbol{\alpha})|^{n-1}$ as follows:

$$|A(\boldsymbol{d}, \boldsymbol{\alpha})|^{n-1}=\sup\{|A(\alpha, \boldsymbol{\alpha})|^{n-1}+1: \boldsymbol{d}(\alpha)\leqq\boldsymbol{d}\}.$$

By induction, the predicate

$$|B(d(\beta), \boldsymbol{\beta})|^{n-1} < |A(d(\alpha), \boldsymbol{\alpha})|^{n-1}$$

is easily seen to be Δ^1_{n-1} uniformly in $\langle A, \alpha, \boldsymbol{\alpha}\rangle$; similarly for $\leqq$. We define $|A(\boldsymbol{\alpha})|^n$ by stipulating that

$$\begin{aligned}
|(\beta)B(\beta, \boldsymbol{\beta})|^n &< |(\alpha)A(\alpha, \boldsymbol{\alpha})|^n \\
&\leftrightarrow \mu(\{d: |B(d, \boldsymbol{\beta})|^{n-1} < |A(d, \boldsymbol{\alpha})|^{n-1}\}) = 1 \\
&\leftrightarrow (\alpha)A(\alpha, \boldsymbol{\alpha}) \mathbin{\&} (\alpha')(\exists\alpha)(d(\alpha') \\
&\qquad \leqq d(\alpha) \mathbin{\&} |B(d(\alpha), \boldsymbol{\beta})|^{n-1} < |A(d(\alpha), \boldsymbol{\alpha})|^{n-1}) \\
&\leftrightarrow (\alpha)A(\alpha, \boldsymbol{\alpha}) \mathbin{\&} (\exists\alpha')(\alpha)(d(\alpha') \leqq d(\alpha) \\
&\qquad \rightarrow |B(d(\alpha), \boldsymbol{\beta})|^{n-1} < |A(d(\alpha), \boldsymbol{\alpha})|^{n-1}).
\end{aligned}$$

If $<$ is well founded, we can simply assign the least ordinals consistent with $<$. But the well-foundedness of $<$ is just a standard fact about ultraproducts with respect to countably additive measures. If $<$ were not well-founded, by DC there would be a sequence $|(\alpha)A_1(\alpha, \boldsymbol{\alpha}_1)|^n > |(\alpha)A_2(\alpha, \boldsymbol{\alpha}_2)|^n > \cdots$. Hence, on a set of measure one, $|A_1(d, \boldsymbol{\alpha}_1)|^{n-1} > |A_2(d, \boldsymbol{\alpha}_2)|^{n-1} > \cdots$.

Reduction for Γ^1_n is now easy. Let $A = \{\alpha: A(\alpha)\}$ and $B = \{\alpha: B(\alpha)\}$, where $A(\alpha)$ and $B(\alpha)$ are Γ^1_n. Let $A' = \{\alpha: A(\alpha) \mathbin{\&} \neg |B(\alpha)|^n < |A(\alpha)|^n\}$ let $B' = \{\alpha: B(\alpha) \mathbin{\&} \neg |A(\alpha)|^n \leqq |B(\alpha)|^n\}$.

We do not know whether AD+DC implies uniformization for Γ^1_n. We conjecture that it does. R. Solovay and the author have shown that AD+DC does yield one new basis theorem: Every nonempty Σ^1_3 set has a Δ^1_4 member. However, our proof does not really need AD, but requires only that a Ramsey cardinal exists.

References

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