ON THE SYMBOL OF A PSEUDO-DIFFERENTIAL OPERATOR

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In [1] Hörmander defines the generalized symbol of a pseudo-differential operator P as a sequence of partially defined maps between function spaces. Our purpose here is to comment on the existence of characteristic polynomial type symbols $\sigma(P)$ and to obtain their composition by introducing a product structure on suitable jet bundles. In particular, this gives the lower order symbol for differential operator on manifold. I express my hearty thanks to J. Bokobza, H. Levine, and A. Unterberger for their indispensable help.

1. **Operation in jet bundle.** Given a compact C^∞ differentiable manifold X, we denote by

$$p_k: J^m(R) \to J^k(R), \qquad m \geqq k,$$

the jet bundle of the trivial bundle $X \times R$ and the canonical projection. Identify the cotangent bundle $T(X)$ as a subbundle of $J^1(R)$ we define the subbundle

$$J_0^k(R) \subseteq J^k(R), \qquad k \geqq 1,$$

as the inverse image by p_1: $J^k(R) \to J^1(R)$ of the nonzero cotangent vector $T_0(X) \subseteq T(X)$. Let E, F, and G be complex vector bundles over X and put

$$J^*(E, F) = \prod_{k=0} \mathrm{Hom}(J_0^{k+1}(R) \oplus J^k(E), F)$$

where "Hom" denotes the space of C^∞ bundle maps which are linear with respect to $J^k(E)$. We shall construct an operation

$$\circ: J^*(E, F) \times J^*(F, G) \to J^*(E, G)$$

as follows. If $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_m, \cdots) \in J^*(E, F)$, $\beta = (\beta_0, \beta_1, \cdots) \in J^*(F, G)$, then

$$\alpha \circ \beta = (\gamma_0, \gamma_1, \cdots, \gamma_r, \cdots) \in J^*(E, G)$$

is given by

$$\gamma_r = \sum_{m+n=r} \beta_n \circ (p_{n+1} \circ p_R \oplus j^n(\alpha_m))$$

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where p_R: $J_0^*(R) \oplus J^*(E) \to J^*(R)$ is the projection, and

$$\overset{n}{j}:\mathrm{Hom}(J_0^{k+1}(R) \oplus \overset{k}{J}(E), F) \to \mathrm{Hom}(J_0^{k+n+1}(R) \oplus \overset{k+n}{J}(E), \overset{n}{J}(F))$$

the nth jet extension map.

THEOREM. *The operation "$\circ$" is well defined, associative, and distributive. Moreover if the α_m (resp. β_n) in α (resp. β) is positive homogeneous of degree $k-m$ (resp. $h-n$) with respect to $J_0^{m+1}(R)$, then $(\alpha \circ \beta)_r$ is positive homogeneous of degree $k+h-r$ (k, h real numbers). In particular, with respect to this operation $J^*(E, E)$ becomes an associative algebra with unity* [2].

2. The symbol homomorphism. Let us recall that a continuous linear map

$$P: C^\infty(E) \to C^\infty(F)$$

between the space of C^∞ sections of complex vector bundles is a pseudo-differential operator of order k in the sense of Hörmander if: for each $f \in C^\infty(E)$, $g \in C^\infty(R)$, such that

(*) $\quad$ $\operatorname{supp} f \subseteq \operatorname{supp}^0 dg$; interior of support of dg

there is a uniform asymptotic expansion [1]

$$e^{-i\lambda g} P(e^{i\lambda g} f) \sim \sum_0^\infty P_j(g, f) \lambda^{k-j}$$

with $P_j(g, f) \in C^\infty(F)$ and $P_0(g, f) \not\equiv 0$. The formal sum $\sum_0^\infty P_j(g, f)$ is the generalized symbol of P.

Now let us denote by

$$\mathcal{P}(E, F) = \sum_k \mathcal{P}_k(E, F)$$

the space of all pseudo-differential operators from the complex vector bundle E to the bundle F over the fixed compact manifold X; $\mathcal{P}_k(E, F)$ those of order k. Then we have

THEOREM. *There exists a unique homomorphism*

$$\sigma: \mathcal{P}(E, F) \to J^*(E, F)$$

satisfying the following conditions:

(1) *If $P \in \mathcal{P}_k(E, F)$, then $\sigma_j(P)$ is positive homogeneous of degree $k-j$ with respect to $J_0^{j+1}(R)$ where $\sigma(P) = (\sigma_0(P), \sigma_1(P), \cdots)$.*

(2) *If $P \in \mathcal{P}(E, F)$, $Q \in \mathcal{P}(F, G)$, then*

$$\sigma(P \circ Q) = \sigma(P) \circ \sigma(Q).$$

(3) *If* $P \in \mathcal{P}_k(E, F)$ *and* $f \in C^\infty(E)$, $g \in C^\infty(\boldsymbol{R})$, *verify the condition* (*), *then the generalized symbol of Hörmander* $P_j(g, f)$ *is equal to the image of* (dg, f) *by the composition*

$$C^\infty(T_0(X)) \times C^\infty(E) \to C^\infty(J_0^{j+1}(\boldsymbol{R}) \oplus J^j(E)) \xrightarrow{\sigma_j(P)} C^\infty(F)$$

restricted on the interior of the support of dg.

(4) *If* P *is a kth order differentiable operator, then* $\sigma_j(P)$ *is defined on* $J^{j+1}(\boldsymbol{R}) \oplus J^j(E)$ *and is zero for* $j > k$. *Moreover the restriction of* $\sigma_0(P)$ *on* $T_0(X) \subseteq J^1(\boldsymbol{R})$:

$$\sigma_0(P): T_0(X) \oplus E \to F$$

is the classical [3] *symbol of the differential operator* P.

REMARK: σ: $\mathcal{P}(E, E) \to J^*(E, E)$ is a homomorphism of algebra with unity. Choose a splitting (e.g. by connections). We obtain an inclusion $T_0(X) \oplus E \hookrightarrow J_0^{j+1}(\boldsymbol{R}) \oplus J^j(E)$; then the restriction of $\sigma_j(P)$ on $T_0(X) \oplus E$ gives the lower order characteristic polynomial of P (e.g. in the case of $\boldsymbol{R}^n$ one gets back the ordinary total characteristic polynomial of a differential operator). Using jet bundles [5] along the fiber, one obtains the same result for a family of operators.

REFERENCES

1. L. Hörmander, *Pseudo-differential operator*, Comm. Pure Appl. Math. **18** (1965), 501–517.

2. J. J. Kohn and L. Nirenberg, *An algebra of pseudo-differential operators*, Comm. Pure Appl. Math. **18** (1965), 269–305.

3. R. Palais, Lecture notes, Brandeis, Univ., Waltham, Mass., 1964.

4. R. T. Seeley, *Singular integrals on compact manifolds*, Amer. J. Math. **81** (1959), 658–690.

5. Weishu Shih, *Fiber cobordism and the index of a family of elliptic differential operators*, Bull. Amer. Math. Soc. **72** (1966), 984–991.

6. A. Unterberger and J. Bokobza, *Les opêrators de Calderôn-Zygmund precisês*, C. R. Acad. Sci. Paris **259** (1964), 1612–1614.

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