ON THE EQUATION $f^n+g^n=1$. II

BY FRED GROSS

Communicated by M. Gerstenhaber, March 13, 1968

In [3], [4] the author proved that

$$f^3 + g^3 = 1 \tag{1}$$

has the solutions

$$f = \frac{1}{2}\left(1 + \frac{\wp'}{\sqrt{3}}\right)/\wp, \qquad g = \frac{1}{2}\left(1 - \frac{\wp'}{\sqrt{3}}\right)/\wp, \tag{2}$$

where the $\wp$-function satisfies

$$(\wp')^2 = 4\wp^3 - 1 \qquad (\text{i.e., } g_2 = 0 \quad \text{and} \quad g_3 = 1). \tag{3}$$

In [3] the author conjectured that all meromorphic solutions of (1) are necessarily elliptic functions of entire functions.

The conjecture was proved by Baker [1]. Baker proved

THEOREM 1. *Any functions $F(z)$ and $G(z)$, which are meromorphic in the plane and satisfy* (1) *have the form*

$$F = f(h(z)), \qquad G = \eta g(h(z)) = \eta f(-h(z)) = f(-\eta^2 h(z)),$$

where f and g are the elliptic functions in (2). *$h(z)$ is an entire function of z and η is a cube-root of unity.*

In this note we give an alternate proof of Theorem 1. In fact we prove

THEOREM 2. *The function f in* (2) *is a uniformizing function of the Riemann surface of*

$$x^3 + y^3 = 1. \tag{4}$$

f maps the whole z-plane in a 1-1 *manner on the Riemann surface of* (4).[1]

PROOF. Since $(1-f^3)^{1/3}$ is a single valued meromorphic function, one can easily verify that either f is a uniformizing function or $f=l(h)$, where l is a uniformizing function and h is nonlinear and entire. The latter, however, is impossible. For otherwise we have $f=l(h)$ and $g=r(h)$, so that f and g and hence $\wp$ and $\wp'$ have h as a common

[1] In the sequel, uniformizing functions are assumed to have this additional property.

right factor (a nonlinear entire function h is a right factor of f if there exists a nonlinear meromorphic l such that $f = l \circ h$). Assume $\wp = l_1(h)$ and $\wp' = l_2(h)$; l_1, l_2 are meromorphic.

It is well known [2] that any uniformizing function of the Riemann surface of (4) is not rational. One can, therefore, verify that l_1 and l_2 must be transcendental. Using this fact one can show [5] that h must be of the form $e^{cz+b}+d$, where b, c and d are constants. One can also show [6] that for any $\wp$ function satisfying (3), h must be a polynomial. Thus $\wp$ and $\wp^1$ cannot have a common right factor. Hence h must be linear and the theorem follows.

We observe that Theorem 1 follows immediately from the fact that a uniformizing function is always determined uniquely up to a linear fractional transformation.

BIBLIOGRAPHY

1. I. N. Baker, *On a class of meromorphic functions*, Proc. Amer. Math. Soc. **17** (1966), 819–822.

2. L. R. Ford, *Automorphic functions*, Chelsea, New York, 1951, pp. 229–240.

3. Fred Gross, *On the equation* $f^n+g^n=1$, Bull. Amer. Math. Soc. **72** (1966), 86–88.

4. ———, *On the equation* $f^n+g^n=1$, Bull. Amer. Math. Soc. **72** (1966), 576.

5. ———, *On factorization of meromorphic functions*, Trans. Amer. Math. Soc. **131** (1968), 215–221.

6. ———, *On factorization of elliptic functions*, Canad. J. Math. **20** (1968), 486–494.

NAVAL RESEARCH LABORATORY, WASHINGTON, D.C.