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FUNCTIONAL INDEPENDENCE OF THETA CONSTANTS

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Communicated by Murray Gerstenhaber, January 20, 1968

1. Introduction and main theorems. If

$$\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} = \begin{bmatrix} \epsilon_1 & \cdots & \epsilon_g \\ \epsilon_1' & \cdots & \epsilon_g' \end{bmatrix}$$

is an even theta g-characteristic ($g \geqq 1$ for this definition but $g \geqq 2$ elsewhere in this note), i.e., a $2 \times g$ matrix with 0, 1 entries, for which $\epsilon \cdot \epsilon' \equiv 0(2)$ (dot is inner product of row g-vectors), and A is a symmetric $g \times g$ complex matrix with positive definite imaginary part, i.e., an element of the Siegel upper half plane $\mathfrak{S}_g$, then the corresponding theta constant is defined by

$$(1) \quad \theta\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} = \sum_n \exp \pi i \{(n + \epsilon/2)A \cdot (n + \epsilon/2) + 2(n + \epsilon/2) \cdot (\epsilon'/2)\},$$

where the sum is over all integral row g-vectors n. There are $2^{g-1}(2^g+1)$ theta constants (explicit dependence on A is suppressed in the notation). These are the "zero values of the first order even theta functions with half-integer characteristics."

It is implicitly assumed, it seems to me, in the literature that the Jacobian of the $2^{g-1}(2^g+1)$ theta constants with respect to the $g(g+1)/2$ independent elements a_{ij}, $i \leqq j$, $i, j = 1, \cdots, g$ of A is generically of maximal rank $g(g+1)/2$ on $\mathfrak{S}_g$, but I have not seen a proof. I present here the sharper, i.e., explicit

[1] Research partially sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, U. S. Air Force, under AFOSR Grant No. AF-AFOSR-1077-66.

THEOREM 1. *Any $g(g+1)/2$ theta constants whose characteristics are obtained by the following prescription are functionally independent for $A \in \mathfrak{S}_g - L$, where L is a fixed analytic set of codimension at least one.*

PRESCRIPTION I. Pick g even g-characteristics as follows: first, pick two even 1-characteristics, i.e., any two of

$$\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \text{ and } \begin{bmatrix}1\\0\end{bmatrix},$$

call them

$$\begin{bmatrix}\epsilon\\\epsilon'\end{bmatrix}, \begin{bmatrix}\mu\\\mu'\end{bmatrix},$$

then for each $1 \leqq i \leqq g$, pick the g-characteristic for which every column is

$$\begin{bmatrix}\epsilon\\\epsilon'\end{bmatrix} \text{ except the } i\text{th which is } \begin{bmatrix}\mu\\\mu'\end{bmatrix}.$$

Then pick $g(g-1)/2$ more as follows: for each index pair (i, j), $i<j$, choose any g-characteristic every column of which is an even 1-characteristic except the ith and jth columns each of which is the unique odd 1-characteristic

$$\begin{bmatrix}1\\1\end{bmatrix}.$$

For example, the $g(g+1)/2$ theta constants

$$\theta\begin{bmatrix}00 & \cdots & 0\\10 & \cdots & 0\end{bmatrix}, \theta\begin{bmatrix}000 & \cdots & 0\\010 & \cdots & 0\end{bmatrix}, \cdots, \theta\begin{bmatrix}0 & \cdots & 00\\0 & \cdots & 01\end{bmatrix},$$

$$\theta\begin{bmatrix}110 & \cdots & 0\\110 & \cdots & 0\end{bmatrix}, \theta\begin{bmatrix}1010 & \cdots & 0\\1010 & \cdots & 0\end{bmatrix}, \cdots, \theta\begin{bmatrix}0 & \cdots & 011\\0 & \cdots & 011\end{bmatrix}$$

satisfy Theorem 1.

THEOREM 2. *The conclusion of Theorem* 1 *remains valid on the Torelli sublocus of $\mathfrak{S}_g$, i.e., the set L intersects that locus in an analytic set $\mathfrak{L}$ of codimension* (*in that locus*) *at least one.*

I remind the reader that the Torelli sublocus of $\mathfrak{S}_g$ is the image of the Torelli space $\mathfrak{J}^g$ under the map ${}^{\#}\pi$ [2, Proposition 5] and consists of the period matrices of the normal abelian integrals of first kind with respect to suitable canonical homology bases on all the Riemann surfaces of genus g. Its complex dimension is $3g-3$.

2. **Proofs of Theorems 1 and 2.** Theorem 1 follows immediately by continuity and other standard arguments from

PROPOSITION 1. *Given a set of* $g(g+1)/2$ *theta constants selected by Prescription* I, *there exists a diagonal matrix* $A^0=\operatorname{diag}(a_{11}^0, \cdots, a_{gg}^0)$ *at which the Jacobian of the set with respect to the* $g(g+1)/2$ *variables* $a_{ii}, a_{ij}, i<j, i, j=1, \cdots, g$ *is not zero.* $A^0 \in \mathfrak{S}_g$.

In the sequel (proof of Proposition 1) I shall assume the a's ordered as follows: $a_{11}, \cdots, a_{gg}$, and then the $a_{ij}, i<j$, lexicographically, and the set of theta constants ordered in the corresponding order as in the example after Theorem 1.

Each set of theta constants per Prescription I leads to an exceptional analytic set. The union of these (finitely many) sets is L.

Theorem 2 results from Proposition 1 and

PROPOSITION 2. *Any neighborhood of a diagonal element of* $\mathfrak{S}_g$ *contains elements of the Torelli sublocus.*

In particular, the Jacobian of Proposition 1 is not zero on all the Torelli sublocus, and so the analytic set L intersects it in an analytic set of lower dimension since the Torelli sublocus is itself an analytic subset of $\mathfrak{S}_g$.

Proposition 2 can be deduced from results in [1]; anyone knowledgeable in this field will accept its validity.

To prove Proposition 1, I need a sequence of lemmata which make use of properties of the elliptic theta functions,

$$(2) \quad \theta\begin{bmatrix}\mu \\ \mu'\end{bmatrix}(u, \tau) = \sum_n \exp \pi i\{(n+\mu/2)^2\tau + 2(n+\mu/2)(u+\mu'/2)\},$$

where the sum is over all integers n and $\operatorname{Im} \tau > 0$. I write

$$\theta\begin{bmatrix}\mu \\ \mu'\end{bmatrix}(0, \tau) = \theta\begin{bmatrix}\mu \\ \mu'\end{bmatrix}(\tau)$$

and use prime for derivation with respect to u of (2).

LEMMA 1. (2) *is even for*

$$\begin{bmatrix}\mu \\ \mu'\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix}, \begin{bmatrix}0 \\ 1\end{bmatrix}, \begin{bmatrix}1 \\ 0\end{bmatrix}$$

and odd for

$$\begin{bmatrix}\mu \\ \mu'\end{bmatrix} = \begin{bmatrix}1 \\ 1\end{bmatrix},$$

all as functions of u. In particular

$$\text{(i)}\ \theta\begin{bmatrix}1\\1\end{bmatrix}(\tau) \equiv 0, \ \textit{and}$$

$$\text{(ii)}\ \theta'\begin{bmatrix}\mu\\\mu'\end{bmatrix}(0,\tau) \equiv 0 \ \textit{for even}\ \begin{bmatrix}\mu\\\mu'\end{bmatrix}.$$

For all τ, $\operatorname{Im}\tau>0$,

$$\text{(iii)}\ \theta'\begin{bmatrix}1\\1\end{bmatrix}(0,\tau) \not\equiv 0, \ \textit{and}$$

$$\text{(iv)}\ \theta\begin{bmatrix}\mu\\\mu'\end{bmatrix}(\tau) \not\equiv 0 \ \textit{for even}\ \begin{bmatrix}\mu\\\mu'\end{bmatrix}.$$

The first part and hence (i), (ii) are well known. (iii) and (iv) follow from standard product expansions.

LEMMA 2. *For any two distinct even theta* 1-*characteristics*

$$\begin{bmatrix}\epsilon\\\epsilon'\end{bmatrix}, \begin{bmatrix}\mu\\\mu'\end{bmatrix}$$

there exists τ^0, $\operatorname{Im}\tau^0>0$, *such that neither the logarithmic derivative of*

$$\theta\begin{bmatrix}\mu\\\mu'\end{bmatrix}(\tau)$$

plus $(g-1)$ *times the logarithmic derivative of*

$$\theta\begin{bmatrix}\epsilon\\\epsilon'\end{bmatrix}(\tau)$$

nor the difference of the logarithmic derivatives is zero at $\tau=\tau^0$.

One sees from (for example)

$$\frac{d}{d\tau}\log\theta\begin{bmatrix}0\\0\end{bmatrix}(\tau) = -\pi i\sum_{m\geq 1}\frac{2mq^{2m}}{1-q^{2m}} + 2\pi i\sum_{m\geq 1}\frac{(2m-1)q^{2m-1}}{1+q^{2m-1}},$$

$$\frac{d}{d\tau}\log\theta\begin{bmatrix}0\\1\end{bmatrix}(\tau) = -\pi i\sum_{m\geq 1}\frac{2mq^{2m}}{1-q^{2m}} - 2\pi i\sum_{m\geq 1}\frac{(2m-1)q^{2m-1}}{1-q^{2m-1}},$$

where $q=e^{\pi i\tau}$, by expanding in series of q ($|q|<1$), that the quotient, a nonconstant meromorphic function, takes values other than $-(g-1)$ or 1.

From (1), one deduces trivially

LEMMA 3. *At* $A = \operatorname{diag}(a_{11}, \cdots, a_{gg}) \in \mathfrak{S}_g$,

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix} = \prod \theta\begin{bmatrix}\epsilon_i\\ \epsilon_i'\end{bmatrix}(a_{ii}),$$

where i *runs from* 1 *to* g *in the product, and*

$$\partial\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix} \Big/ \partial a_{ii} = \prod{}' \theta\begin{bmatrix}\epsilon_j\\ \epsilon_j'\end{bmatrix}(a_{jj}) d\theta\begin{bmatrix}\epsilon_i\\ \epsilon_i'\end{bmatrix}(a_{ii})/d\tau,$$

and the primed product omits $j = i$.

From Prescription I and Lemma 1, (i) one has

COROLLARY 1. *Under the same hypothesis, the partials of the last (see remark after Proposition* 1) $g(g-1)/2$ *theta constants in Proposition* 1 *with respect to the first* g *(the diagonal) variables are all zero.*

Factoring and using the rules of determinants one has

COROLLARY 2. *The upper left hand* $g \times g$ *subdeterminant of the Jacobian in Proposition* 1 *equals* $C\Delta$ *where* Δ *is the determinant whose* (i, i) *entry is*

$$d \log \theta\begin{bmatrix}\mu\\ \mu'\end{bmatrix}(a_{ii})/d\tau$$

and whose (i, j) *entry,* $i \neq j$, *is*

$$d \log \theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(a_{jj})/d\tau, \quad i, j = 1, \cdots, g,$$

and

$$C = \left\{\prod_i \theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(a_{ii})\right\}^{g-1} \prod_j \theta\begin{bmatrix}\mu\\ \mu'\end{bmatrix}(a_{jj}).$$

COROLLARY 3. *If* $A^0 = \operatorname{diag}(\tau^0, \cdots, \tau^0)$ *where* τ^0 *is taken from Lemma* 2, *then the determinant of Corollary* 2 *is not zero.*

For by Lemma 1, (iv) $C \neq 0$, and the vanishing of Δ at A^0 would imply a linear dependence of its rows,

$$c_1 \frac{d}{d\tau} \log \theta\begin{bmatrix}\mu\\ \mu'\end{bmatrix}(\tau^0) + (c_2 + \cdots + c_g) \frac{d}{d\tau} \log \theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\tau^0) = 0,$$

$$\cdots$$

$$(c_1 + \cdots + c_{g-1}) \frac{d}{d\tau} \log \theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\tau^0) + c_g \frac{d}{d\tau} \log \theta\begin{bmatrix}\mu\\ \mu'\end{bmatrix}(\tau^0) = 0,$$

or, if one adds,

$$(c_1 + \cdots + c_g)\left\{\frac{d}{d\tau}\log\theta\begin{bmatrix}\mu\\ \mu'\end{bmatrix}(\tau^0) + (g-1)\frac{d}{d\tau}\log\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\tau^0)\right\} = 0.$$

If $c_1 + \cdots + c_g \neq 0$, Lemma 2 is violated. If, say,

$$c_i = -(c_1 + \cdots c_{i-1} + c_{i+1} + \cdots + c_g),$$

then the ith line of the preceding equations shows that Lemma 2 is again violated.

LEMMA 4. *At* $A = \mathrm{diag}(a_{11}, \cdots, a_{gg}) \in \mathfrak{S}_g$, $i < j$,

$$\partial\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix} \Big/ \partial a_{ij} = (1/2\pi i)\prod_k{}'' \theta\begin{bmatrix}\epsilon_k\\ \epsilon_k'\end{bmatrix}(a_{kk})\theta'\begin{bmatrix}\epsilon_i\\ \epsilon_i'\end{bmatrix}(0, a_{ii})\theta'\begin{bmatrix}\epsilon_j\\ \epsilon_j'\end{bmatrix}(0, a_{jj})$$

where double prime means i and j omitted.

The proof follows by differentiating (1) and (2), setting $A = \mathrm{diag}(a_{11}, \cdots, a_{gg})$, and comparing.

Proposition 1 now follows for

$$\overset{0}{A} = \mathrm{diag}(\overset{0}{a}_{11}, \cdots, \overset{0}{a}_{gg}) = \mathrm{diag}(\overset{0}{\tau}, \cdots, \overset{0}{\tau}),$$

where τ^0 comes from Lemma 2. Lemma 4 and Lemma 1, (i), (ii), (iii), and (iv) imply that all the partials of the first g theta constants with respect to the last $g(g-1)/2$ variables are zero and that, of all the partials of the last $g(g-1)/2$ theta constants with respect to the same variables, those and only those on the diagonal are not zero. Combining this with Corollaries 1, 2, and 3 finishes the proof.

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