References

1. G. Brauer, Research Problem: *Sets of convergence of exponential series*, Bull. Amer. Math. Soc. **72** (1966), p. 470.
2. L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, New York, 1960.
3. N. Dunford and J. T. Schwartz, *Linear operators*. I, Interscience, New York, 1958.
4. V. M. Darevsky, *On intrinsically perfect methods of summation*, Bull. Acad. Sci. URSS Ser. Mat. (Izv. Akad. Nauk SSSR Ser. Mat.) **10** (1946), 97–104.

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THE UNION OF FLAT $(n-1)$-BALLS IS FLAT IN R^n

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THEOREM.[2] *Let β_1^{n-1} and β_2^{n-1} be two locally flat $(n-1)$-balls in R^n with $\beta_1 \cap \beta_2 = \partial\beta_1 \cap \partial\beta_2 = \beta^{n-2}$, where β^{n-2} is an $(n-2)$-ball which is locally flat in $\partial\beta_1$ and $\partial\beta_2$. Then $\beta_1 \cup \beta_2$ is a flat $(n-1)$-ball in R^n.*

This result has been announced by Černavskiĭ [1], but only for $n \geqq 5$ since his outlined proof uses engulfing. Our proof avoids engulfing and works for all n; a thorough knowledge of Cantrell and Lacher's version (see [2, §§4 and 5]) of Černavskiĭ's theorem is necessary to understand our proof.

We also have another proof of the following corollary which appears in [4].

COROLLARY. *Let $g: M^{n-1} \rightarrow N^n$ be an imbedding of an $(n-1)$-manifold into an n-manifold which is locally flat except on a set E. If $n > 3$, then E contains no isolated points (see [3] for the same result when M and N are spheres).*

PROOF. Let C be a neighborhood of an isolated point p in M which is homeomorphic to an $(n-1)$-ball, with g locally flat on $C-p$. Then split C into $(n-1)$-balls C_1 and C_2 so that $C = C_1 \cup C_2$ and $C_1 \cap C_2$ is an $(n-2)$-ball containing p. g is locally flat on C_1 and C_2 except at the point p on their boundaries. Then, since $n > 3$, g is flat on all of C_1 and C_2 by [5]. It follows from the theorem that $C_1 \cup C_2 = C$ is flat, so E has no isolated points.

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[2] *Added in proof.* Černavskiĭ has independently proven this theorem by similar methods.

Let R^n be Euclidean n-space, B^n be the unit n-ball, and R^k be imbedded in R^n as $R^k=\{x\in R^n \mid x_{k+1}= \cdots =x_n=0\}$. We will coordinatize R^n by using $R^n=R^{n-2}\times R^2$ with polar coordinates on R^2. Thus points of R^n will be triples (z, r, θ) with $z\in R^{n-2}$, $r\geqq 0$, and $\theta\in R$ and with the convention that $(0, r, 0)$ is a point on the positive x_{n-1}-axis and $(0, r, \pi/2)$ is a point on the positive x_n-axis. Let $H_\phi=\{(z, r, \theta)\in R^n \mid \theta=\phi\}$ and $D_\phi=H_\phi\cap B^n$. Note that $D_\pi\cup D_0=B^{n-1}$ and $D_\pi\cap D_0=B^{n-2}$. Let $W(\theta_1, \theta_2)$ be the wedge $\{(z, r, \theta) \mid \theta_1\leqq\theta\leqq\theta_2\}$ and $\tilde{W}(\theta_1, \theta_2)=W(\theta_1, \theta_2)\cap B^n$.

Proof of Theorem. Suppose β_1 and β_2 are given by imbeddings $f_1\colon D_\pi\to R^n$ and $f_2\colon D_0\to R^n$. Since β^{n-2} is locally flat in $\partial\beta_1$ and $\partial\beta_2$, the closures of $\partial\beta_1-\beta^{n-2}$ and $\partial\beta_2-\beta^{n-2}$ are homeomorphic to $(n-1)$-balls. Then we may assume that $f_1(D_\pi)\cap f_2(D_0)=f_1(B^{n-2})=f_2(B^{n-2})=\beta^{n-2}$.

Since locally flat imbeddings of balls are flat, f_1 and f_2 extend to imbeddings of R^n into R^n (still called f_1 and f_2). We can require that the extensions are chosen so that $f_1(H_\pi)\cap f_2(D_0)=\beta^{n-2}$ and $f_2(B^n)\subset f_1(R^n)$. Then it suffices to show that $D_\pi\cup f_1^{-1}f_2(D_0)$ is locally flat. Let $f=f_1^{-1}f_2$.

Since $f(D_0)\cap H_\pi=B^{n-2}$, we can assume that $f(D_0)\subset W(0, \pi/4)$ by rotating $f(D_0)$ around R^{n-2} and away from H_π while fixing H_π. Then, in the coordinates of $f(B^n)$, we can rotate $f(D_\pi)$ close to $f(D_0)$, so we may as well assume that $f(D_\pi)\subset W(0, \pi/4)$ and lies between $H_{\pi/4}$ and $f(D_0)$ (see Figure 1).

Let $\tilde{h}\colon R^n-\operatorname{int} H_0\to R^n-\operatorname{int} W(0, \pi/2)$ be the obvious homeomorphism which takes the wedge $W(0, \pi)-\operatorname{int} H_0$ onto $W(\pi/2, \pi)-\operatorname{int} H_{\pi/2}$ and fixes $\operatorname{int} W(\pi, 2\pi)$. The set $W(0, \pi)\cap f(B^n)$ is separated

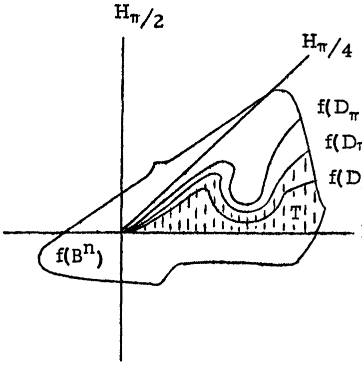

Figure 1

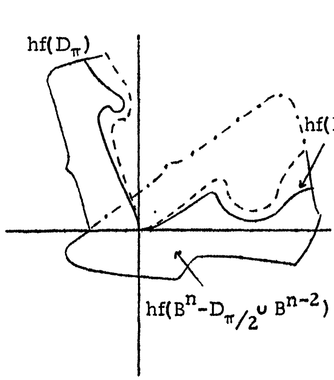

Figure 2

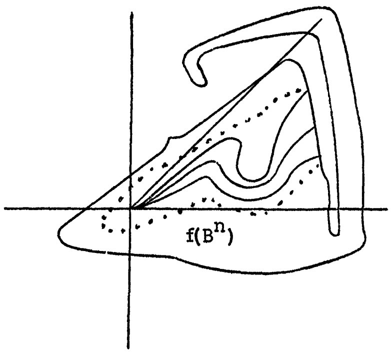

FIGURE 3

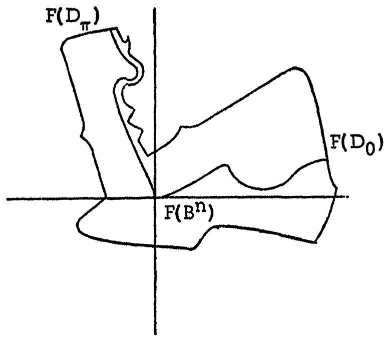

FIGURE 4

into two sets by $f(D_{\pi/2})$; let T denote the set containing $f(D_0)$. Then (see Figure 2) define an imbedding $h\colon f(B^n - D_{\pi/2} \cup B^{n-2}) \to R^n$ by

$$\begin{aligned} h(f(x)) &= f(x) \quad \text{if } f(x) \in T, \\ &= \tilde{h}f(x) \quad \text{if } f(x) \notin T. \end{aligned}$$

To ensure that h is an imbedding it may be necessary to trim away part of $f(B^n)$, still leaving a "ball-neighborhood" of $f(D_0)$ (in Figure 3, restricting to the dotted ball would eliminate the annoying feelers). Note that $hf = f$ on D_0 and $hf(D_\pi) \subset W(\pi/2, \pi)$.

We need to extend $hf \mid \tilde{W}(\pi, 2\pi)$ to an imbedding of B^n into R^n. We can assume that for some $\epsilon > 0$, $f(D_{2\pi-\epsilon}) \subset W(0, \pi/2)$, so then $hf = f$ on $D_{2\pi-\epsilon}$. Let g_1 be the homeomorphism of $B^n - D_{\pi/2} \cup B^{n-2}$ which fixes points outside $\tilde{W}(3\pi/4, 2\pi)$ and moves $D_{2\pi-\epsilon}$ to D_π. Let $g_2\colon hf(\tilde{W}(3\pi/4, 2\pi - \epsilon)) \to hf(\tilde{W}(3\pi/4, \pi))$ be the homeomorphism defined by $g_2 = hfg_1(hf)^{-1}$. Now define an imbedding $g\colon f(\tilde{W}(0, 2\pi - \epsilon) \to R^n$ by

$$\begin{aligned} g(x) &= g_2(x) \quad \text{if } x \in hf(\tilde{W}(3\pi/4, 2\pi - \epsilon)), \\ &= x \qquad \text{otherwise.} \end{aligned}$$

To make sure that g is well defined, it may be necessary to again shrink $f(B^n)$ towards B^{n-2} so that $\operatorname{int} f(\tilde{W}(0, 2\pi - \epsilon)) \cap \partial hf(\tilde{W}(3\pi/4, 2\pi - \epsilon)) \subset hf(D_{3\pi/4})$. Let $i\colon \tilde{W}(0, \pi) \to \tilde{W}(0, 2\pi - \epsilon)$ and note that $gfi = hf$ on D_π. Then (see Figure 4), we can piece together gfi and hf to get an imbedding $F\colon B^n \to R^n$; specifically, let

$$\begin{aligned} F(x) &= gfi(x) \quad \text{if } x \in \tilde{W}(0, \pi), \\ &= hf(x) \quad \text{if } x \in \tilde{W}(\pi, 2\pi). \end{aligned}$$

$F = f$ on D_0, so $F(D_0) \subset W(-\pi/2, \pi/2)$, and $F(D_\pi) = hf(D_\pi)$

$\subset W(\pi/2, 3\pi/2)$. Thus $F(B^{n-1})$ is "transverse" to $H_{\pi/2} \cup H_{3\pi/2}$, and that is the key to the proof. It allows us to find an isotopy making $F(D_\theta)$ tangent to H_θ at B^{n-2} for all θ. This isotopy is constructed in the latter part of the proof of Lemma 5.2 of [2]. Then a homeomorphism of R^n can be constructed which fixes D_π and takes $F(D_0)$ to D_0 (see the proof of Theorem 6.1 in [2]). Thus $(R^n, \beta_1 \cup \beta_2)$ is pairwise homeomorphic to $(R^n, D_\pi \cup D_0)$, finishing the proof.

References

1. A. V. Černavskiĭ, *Singular points of topological imbeddings of manifolds and the union of locally flat cells*, Soviet Math. Dokl. **7** (1966), 433–436.

2. J. C. Cantrell and R. C. Lacher, *Some conditions for manifolds to be locally flat*, (to appear).

3. T. Hutchinson, *Two-point spheres are flat*, Notices Amer. Math. Soc. **14** (1967), 364, Abstract 644–18.

4. R. C. Kirby, *On the set of non-locally flat points of a submanifold of co-dimension one*, Ann. of Math.

5. R. C. Lacher, *Locally flat strings and half-strings*, Proc. Amer. Math. Soc. **18** (1967), 299–304.

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