A GEOMETRIC INTERPRETATION OF THE KÜNNETH FORMULA FOR ALGEBRAIC K-THEORY

BY F. T. FARRELL AND W. C. HSIANG[1]

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1. Introduction. A Künneth Formula for Whitehead Torsion and the algebraic K_1 functor was derived in [1], [2]. The formula reads as follows. Let A be a ring with unit and $A[T]$ be the finite Laurent series ring over A. Then, there is an isomorphism $K_1A[T] \cong K_1A \oplus K_0A \oplus L_1(A, T)$ where $L_1(A, T)$ are generated by the images in $K_1A[T]$ of all $I + (t^{\pm 1} - 1)\beta$, with β a nilpotent matrix over A. On the other hand, a group $C(A, \alpha)$ was introduced by one of the authors in his thesis [3], [4] in order to study the obstruction to fibring a manifold over S^1. The group $C(A, \alpha)$ is the Grothendieck group of finitely generated projective modules over A with α semilinear nilpotent endomorphisms where α is a fixed automorphism of A. The structure of $C(A, \alpha)$ suggests its close relation with the above Künneth Formula. This relation gradually became clear to us after we wrote the joint paper [5]. Since fibring a manifold over S^1 is a codimension one embedding problem, one expects a good geometric interpretation of the above formula in terms of the obstruction to finding a codimension one submanifold.

In this note, we announce this interpretation which will make the relationship of [1], [2] and [3], [4], [5] even clearer. In order to put our geometric theorems in a more natural setting, we generalize the Künneth Formula to $K_1A_\alpha[T]$ where α is an automorphism of A and $A_\alpha[T]$ is the α-twisted finite Laurent series ring over A. This generalization is given in §2.

This note is an attempt to understand more about nonsimply connected manifolds and the functors K_0, K_1. A systematic account will appear later. We are indebted to W. Browder for calling our attention to the codimension one embedding problem.

2. The Künneth Formula for $K_1A_\alpha[T]$. Let A be a ring with unit. The α-twisted polynomial ring $A_\alpha[t]$ is defined as follows. Additively, $A_\alpha[t] = A[t]$. Multiplicatively, for $f = at^n$, $g = bt^m$ two monomials, $f \cdot g = a\alpha^n(b)t^{n+m}$. Similarly, we define $A_\alpha[T] = A_\alpha[t, t^{-1}]$. The inclusion $i\colon A_\alpha[t] \subset A_\alpha[T]$ induces the exact sequence [2], [6]

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$$(1)\qquad K_1A_\alpha[t] \xrightarrow{i_*} K_1A_\alpha[T] \xrightarrow{q} K_1\Phi(i) \xrightarrow{\partial} K_0A_\alpha[t] \xrightarrow{i_*} K_0A_\alpha[T].$$

The group $K_1\Phi(i)$, and the homomorphisms q, ∂ are described as follows. An element in $K_1\Phi(i)$ is represented by a class $[P, a, Q]$ where P, Q are finitely generated projective modules over $A_\alpha[t]$ and

$$a: A_\alpha[T]\otimes_{A_\alpha[t]} P \rightarrow A_\alpha[T]\otimes_{A_\alpha[t]} Q$$

is an isomorphism. Let $[(A_\alpha[T])^n, a]$ represent an element in $K_1A_\alpha[T]$. Then $q[(A_\alpha[T])^n, a] = [(A_\alpha[t])^n, a, (A_\alpha[t])^n]$. This definition makes sense, since $(A_\alpha[T])^n = A_\alpha[T]\otimes_{A_\alpha[t]}(A_\alpha[t])^n$. For $[P, a, Q]$ in $K_1\Phi(i)$, $\partial[P, a, Q] = [P]-[Q]$. Now, let us recall the group $C(A, \alpha)$ introduced in [3], [4]. $C(A, \alpha)$ is the abelian group generated by all the isomorphism classes $[P, f]$ where P is a finitely generated projective module over A with an α semilinear nilpotent endomorphism f, modulo all the relations $[P_2, f_2] = [P_1, f_1] + [P_3, f_3]$ for all the short exact sequences $0\rightarrow(P_1, f_1)\rightarrow(P_2, f_2)\rightarrow(P_3, f_3)\rightarrow 0$. The "Forgetting Functor" by throwing away the endomorphism defines a homomorphism

$$(2)\qquad j: C(A, \alpha) \rightarrow K_0(A) \xrightarrow{\alpha_*-\mathrm{id}} K_0A \xrightarrow{h} K_0A_\alpha[t],$$

where h is induced by inclusion. Let $\tilde{C}(A, \alpha)$ be the subgroup of $C(A, \alpha)$ generated by $[A^n, a]-[A^n, 0]$. It was proved in [3], [4] that we have the natural decomposition $C(A, \alpha) = \tilde{C}(A, \alpha)\oplus K_0A$. Let us define $\bar{C}(A, \alpha) = \tilde{C}(A, \alpha)\oplus\tilde{K}_0A$, and let $C(A, \alpha)^\alpha$, $\bar{C}(A, \alpha)^\alpha$ be the subgroups of $C(A, \alpha)$ and $\bar{C}(A, \alpha)$, respectively, consisting of elements invariant under α. Now, let us consider the following construction. Let $[P, a, Q]$ be an element in $K_1\Phi(i)$. Since

$$P = A_\alpha[t]\otimes_{A_\alpha[t]} P \subset A_\alpha[T]\otimes_{A_\alpha[t]} P$$

and

$$Q = A_\alpha[t]\otimes_{A_\alpha[t]} Q \subset A_\alpha[T]\otimes_{A_\alpha[t]} Q,$$

we can find t^n $(n\geqq 0)$ such that $L_{t^n}\circ a(P)\subset Q$ where L_{t^n} is the left multiplication[2] by t^n.

THEOREM 1. (a) *$M(L_{t^n}\circ a) = Q/L_{t^n}\circ a(P)$ is a finitely generated projective module over A and L_t defines an α semilinear nilpotent endomorphism t on $M(L_{t^n}\circ a)$.*

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[2] $L_{t^n}\circ a$ is α^n semilinear.

(b) *If we define* $\chi: K_1\Phi(i) \to C(A, \alpha)$ *by setting*

$$\chi[P, a, Q] = [M(L_{t^n} \circ a), t] - [P/L_{t^n}(P), t],$$

then χ *is an isomorphism. Moreover, the following triangle is commutative.*

$$\begin{array}{ccc} K_1\Phi(i) & \xrightarrow{\partial} & K_0A_\alpha[t] \\ & \searrow\chi \quad \nearrow j & \\ & C(A, \alpha) & \end{array} \tag{3}$$

Therefore, the sequence (1) becomes

$$K_1A_\alpha[t] \xrightarrow{i_*} K_1A_\alpha[T] \xrightarrow{p} C(A, \alpha) \xrightarrow{j} K_0A_\alpha[t] \xrightarrow{i_*} K_0A_\alpha[T] \tag{4}$$

where $p = \chi \circ q$. The typical example of $A_\alpha[T]$ comes as follows. Let $G \odot_\alpha Z$ be a split extension $1 \to G \to G \odot_\alpha Z \to Z \to 1$, such that a generator t of Z acts on G as an automorphism α of G. Then $\mathbf{Z}(G \odot_\alpha Z) = \mathbf{Z}(G)_\alpha[T]$. Let $G \odot_\alpha Z^+$ be the induced split extension of G by the semigroup of nonnegative integers Z^+, and let us write

$$\text{Wh } G \odot_\alpha Z^+ = K_1\mathbf{Z}(G \odot_\alpha Z^+)/J$$

where J is the subgroup generated by $\{\pm 1\}$ and $\{G\}$. The inclusion $i': G \subset G \odot_\alpha Z^+$ induces a homomorphism $i'_*: \text{Wh}G \to \text{Wh}G \odot_\alpha Z^+$. Let $p_1: K_1\mathbf{Z}(G \odot_\alpha Z) = K_1\mathbf{Z}(G)_\alpha[T] \to C(A, \alpha)$ be the homomorphism defined as p except that we consider the inclusion $K_1\mathbf{Z}(G)_\alpha[t^{-1}] \subset K_1\mathbf{Z}(G)_\alpha[T]$ instead. The composite of homomorphisms

$$K_1\mathbf{Z}(G \odot_\alpha Z^+) = K_1\mathbf{Z}(G)_\alpha[t] \to K_1\mathbf{Z}(G \odot_\alpha Z) = K_1\mathbf{Z}(G)_\alpha[T] \xrightarrow{p} C(A, \alpha)$$

induces a homomorphism $p': \text{Wh}G \odot_\alpha Z^+ \to \tilde{C}(A, \alpha)$.

LEMMA 1. *The following sequence is short exact:*

$$0 \to \text{Wh } G \xrightarrow{i'_*} \text{Wh } G \odot_\alpha Z^+ \xrightarrow{p'} \tilde{C}(A, \alpha) \to 0.$$

Let I and I_1 be the subgroups of $K_1A_\alpha[t]$ and $\text{Wh}G$, respectively, generated by $x - \alpha_* x$ for $x \in K_1A_\alpha[t]$ or $\text{Wh}G$, respectively. Using Lemma 1, I_1 can be considered as a subgroup of $\text{Wh}G \odot_\alpha Z^+$.

THEOREM[3] 2 (KÜNNETH FORMULA FOR $K_1A_\alpha[T]$ OR $\text{Wh}G \odot_\alpha Z$). *The following two sequences are exact:*

[3] C. T. C. Wall has proven this theorem independently.

$$(5)\qquad \begin{aligned} &K_1A_\alpha[t]/I \xrightarrow{i_*} K_1A_\alpha[T] \xrightarrow{p} C(A,\alpha)^\alpha \to 0,\\ &\text{Wh } G\odot_\alpha Z^+/I_1 \xrightarrow{i_*} \text{Wh } G\odot_\alpha Z \xrightarrow{p} \overline{C}(\boldsymbol{Z}(G),\alpha)^\alpha \to 0. \end{aligned}$$

REMARKS. (a) For $\alpha=\text{id}$, the sequences of (5) are split short exact and $I=0$, $I_1=0$, $C(A,\alpha)^\alpha=C(A,\text{id})$, $\overline{C}(\boldsymbol{Z}(G),\alpha)^\alpha=\overline{C}(\boldsymbol{Z}(G),\text{id})$. These sequences together with those for the inclusions $A_\alpha[t^{-1}]\subset A_\alpha[T]$, $G\odot_\alpha Z^-\subset G\odot_\alpha Z$ (where Z^- is the semigroup of nonpositive integers) lead to the Künneth Formula of [1], [2] mentioned in the introduction.

(b) $\tilde{C}(A,\alpha)^\alpha$ is always equal to $\tilde{C}(A,\alpha)$.

(c) When $A=\boldsymbol{Z}(G)$ for G a finitely presented group, the sequences (5) are short exact by a geometric proof. We believe that they are always short exact.

(d) For $\alpha=\text{id}$, the Künneth Formula is a generalization of Bott's periodicity [1], [2].

3. **Homotopic interpretation of p: $K_1A_\alpha[T]\to C(A,\alpha)$.** Let $\boldsymbol{C}$: $C_n\to C_{n-1}\to\cdots\to C_1\to C_0\to 0$ be a based free finitely generated chain complex over $A_\alpha[t]$. Then the basis of $\boldsymbol{C}$ induces a basis for $\boldsymbol{C}'=A_\alpha[T]\oplus_{A_\alpha[t]}\boldsymbol{C}$.

LEMMA 2. *Let $\boldsymbol{C}$ and $\boldsymbol{C}'$ be given as above. Assume that $\boldsymbol{C}'$ is acyclic and*

$$H_i(\boldsymbol{C}) = 0 \quad i\neq s \quad \text{for } 0\leqq s\leqq n,$$

$$\operatorname*{Proj\,dim}_{A_\alpha[t]} H_s(\boldsymbol{C}) \leqq 1.$$

Then $[H_s(\boldsymbol{C}),t]$ is in $C(A,\alpha)$ and $p(\tau(\boldsymbol{C}'))=(-1)^s[H_s(\boldsymbol{C}),t]$ where $\tau(\boldsymbol{C}')\in K_1A_\alpha[T]$ is the torsion of $\boldsymbol{C}'$.

Now, let $(K;K_1,K_2)$ be a triad of finite CW-complexes with $\Pi_1K=G\odot_\alpha Z$. Suppose that Π_1K_2 is the normal subgroup G under the inclusion. Suppose that we can lift K_2 into the covering space X of K corresponding to G such that K_2 divides X into A and B with $t(A)\subset A$ where t now stands for a generator of the ∞-cyclic group of covering transformations. Assume that K_1 is a deformation retract of K. Set Y to be the portion of X over K_1. Assume further that (a) $H_i(A,A\cap Y;\boldsymbol{Z}(G))=0$ for $i\neq s$, and (b) $\text{Proj dim}_{\boldsymbol{Z}(G)_\alpha[t]}H_s(A,A\cap Y;\boldsymbol{Z}(G))\leqq 1$. Then $H_s(A,A\cap Y;\boldsymbol{Z}(G))$ is a finitely generated projective module over $\boldsymbol{Z}(G)$, and the covering transformation t induces an α semilinear nilpotent endomorphism on $H_s(A,A\cap Y;\boldsymbol{Z}(G))$. Denote the corresponding element in $\overline{C}(\boldsymbol{Z}(G),\alpha)$ by $[H_s,t]$.

THEOREM 3 (HOMOTOPIC INTERPRETATION OF p). *Let $(K; K_1, K_2)$ be given as above, and let $\tau(K, K_1) \in \mathrm{Wh}\Pi_1 K_1 = \mathrm{Wh}G \odot_\alpha Z$ be the torsion of the pair (K, K_1). Then*

$$p\tau(K, K_1) = (-1)^s[H_s, t]. \tag{6}$$

Now, let $f: K \to L$ be a homotopy equivalence of finite CW-complexes. Suppose that $\Pi_1 L = G \odot_\alpha Z$ and L_1 is a subcomplex of L with $\Pi_1 L_1 = G$ under the inclusion. Let X be the covering space of L corresponding to G. Suppose that a lifting of L_1 into X divides X into A_L and B_L such that $t(A_L) \subset A_L$ for t a generator of the group of covering transformations. Let Y be the corresponding covering space of K and $f_1: Y \to X$ be a covering map. Set $K_1 = f^{-1}(L_1)$, $A_K = f_1^{-1}(A_L)$, and $B_K = f_1^{-1}(B_L)$. Assume that (a) $f_*: H_i(A_K; \mathbf{Z}(G)) \to H_i(A_L; \mathbf{Z}(G))$ is always epimorphic, (b) f_* is monomorphic except for $i = s$, (c) Proj dim $\mathbf{Z}_{(G)_\alpha[t]}$ Ker $f_* \leq 1$. Then Ker f_* is a finitely generated projective module over $\mathbf{Z}(G)$ and the covering transformation t induces an α semilinear endomorphism on Ker f_*. Denote its class in $\overline{C}(\mathbf{Z}(G), \alpha)$ by $[\mathrm{Ker} f_*, t]$.

COROLLARY 1. *Let $f: K \to L$ be given as above and $\tau(f) \in \mathrm{Wh}G \odot_\alpha Z$ be the torsion of f. Then $p(\tau(f)) = (-1)^s[\mathrm{Ker} f_*, t]$.*

4. Geometric interpretation of the Künneth Formula. Now, let M_1 be an n-dim closed manifold[4] with $\Pi_1 M_1 = G \odot_\alpha Z$. Let N_1 be an $(n-1)$-dim closed submanifold of M_1 such that $\Pi_1 N_1 = G$ under the inclusion. Let M_2 be another n-dim closed manifold and $f: M_2 \to M_1$ be a homotopy equivalence. We ask what is *the obstruction* $O(f)$ to finding an $(n-1)$-dim submanifold N_2 in M_2 and a map $g: (M_2, N_2) \to (M_1, N_1)$ such that (a) g is a homotopy equivalence, (b) $f^{-1}(N_1) = N_2$, (c) the induced map $g: M_2 \to M_1$ is homotopic to the original map f. $O(f)$ is called the obstruction to the splitting of f with respect to N_1. Now, suppose that such N_2 and g exist. Cut M_1 and M_2 along N_1 and N_2, respectively, to form manifolds with boundaries $\overline{M}_1$ and $\overline{M}_2$ such that both boundaries of $\overline{M}_1$ or $\overline{M}_2$ are N_1 or N_2, respectively. Let X and Y be the covering space of M_2 and M_1 corresponding to G, respectively. We can lift $\overline{M}_2$ and $\overline{M}_1$ into X and Y, respectively, and find a covering map $f_1: X \to Y$ which sends the lifted image of $\overline{M}_2$ into that of $\overline{M}_1$. Denote such a map[5] by $h: \overline{M}_2 \to \overline{M}_1$. h induces a map $h|N_2: N_2 \to N_1$. They are homotopy equivalence, and hence the tor-

[4] We only state our results for closed manifolds for exposition simplicity. We can generalize our results to a more general setting.

[5] Such a map is not unique (cf. Theorem 4(b)).

sions are defined. Set $\tau_1=\tau(h)-\tau(h|N_2)\in \mathrm{Wh}G$. We ask what is $i_*\tau_1\in \mathrm{Wh}G\odot_\alpha Z$.

THEOREM 4 (GEOMETRIC INTERPRETATION OF THE KÜNNETH FORMULA). *Assume that* (M_1, N_1), $f: M_2\to M_1$ *are given as above and* $n\geqq 6$. *Then we have the following conclusions:* (a) *The obstruction* $O(f)$ *to the splitting of* f *with respect to* N_1 *is equal to* $p\tau(f)$ *where* $\tau(f)\in \mathrm{Wh}G\odot_\alpha Z=\mathrm{Wh}\Pi_1 M_2$ *is the torsion of* f. (b) *If* $O(f)=p\tau(f)$ *vanishes, then* $i_*\tau_1=\tau(f)\in \mathrm{Wh}G/I_1\subset \mathrm{Wh}G\odot_\alpha Z$. *Moreover, for every element* τ_1 *in the coset* $\tau(f)$ *of* $\mathrm{Wh}G/I_1$, *we can find some* g': $(M_2, N_2')\to(M_1, N_1)$ *and a lifting* h': $\overline{M}_2'\to\overline{M}_1$ *such that* $\tau_1=\tau(h')-\tau(h'|N_2')$.

COROLLARY 2 (SPLITTING A SIMPLE HOMOTOPY EQUIVALENCE). *Under the same assumptions of Theorem* 4, *if* f *is a simple homotopy equivalence, then* $O(f)$ *vanishes, and hence* f *is splittable with respect to* N_1.

COROLLARY 3 (PRODUCT FORMULA FOR $\tilde{C}(A, \alpha)$). *Let* (M_1, N_1), f: $M_2\to M_1$ *be given as in Theorem* 4 *and let* L *be a fixed closed manifold. Hence,* $L\times N_1\subset L\times M_1$ *is a codimension one embedding and* $(id\times f)$: $L\times M_1\to L\times M_2$ *is a homotopy equivalence. Then the obstruction* $O(id\times f)$ *to splitting* $(id\times f)$ *with respect to* $(L\times N_1)$ *is equal to* $\chi(L)\cdot j_*O(f)$ *where* $\chi(L)$ *is the Euler characteristic of* L *and* j_*: $\tilde{C}(\Pi_1N_1,\alpha)\to\tilde{C}(\Pi_1(L)\times\Pi_1(N_1), id\times\alpha)$ *is induced by the inclusion* $N_1\subset L\times N_1$.

COROLLARY 4 (FIBRING A SIMPLE HOMOTOPY EQUIVALENCE). *Suppose that* M_1^n $(n\geqq 6)$ *fibres over* S^1 *with respect to* f_1: $M_1^n\to S^1$. *Let* g: $M_2^n\to M_1^n$ *be a simple homotopy equivalence. Then* M_1^n *fibres over* S^1 *with respect to* $f_2=f_1\circ g$: $M_2^n\to S^1$.

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YALE UNIVERSITY