ON WEAK MIXING METRIC AUTOMORPHISMS[1]

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Let $(X, \mathfrak{A}, P)$ be a separable probability space and T a metric automorphism of the space onto itself, i.e. T is, except for null sets, a one to one invertible map from X to X such that $T^{-1}(\mathfrak{A})=\mathfrak{A}$ and $P(T^{-1}A)=P(A)$ for all $A\in\mathfrak{A}$.

There are three standard types of mixing for metric automorphisms, namely ergodic, weak mixing, and strong mixing [2]. It is known [1] that an automorphism is ergodic if and only if for all sets A and B from $\mathfrak{A}$ which have nonzero measure there exists a positive integer n such that $P(T^nA\cap B)>0$. In this note we show that a similar condition which we call property W is necessary and sufficient for weak mixing.

PROPERTY W. For every two sets A and B of strictly positive measure there exists a subset K of the positive integers with density zero such that for all $k\notin K$, $P(T^kA\cap B)>0$.

LEMMA. *If a metric automorphism T satisfies property* W *then it is strongly ergodic, i.e. every nonzero integral power of T is ergodic.*

PROOF. Let m be a given positive integer and A and B two sets of positive measure. Let K denote the set of density zero associated with A and B by property W. Denote by M the set of integers mk where k runs over the positive integers. Since the upper density of M is positive, M is not contained in K and there exists $mk\notin K$. Thus $P(T^{mk}A\cap B)=P((T^m)^kA\cap B)>0$ and T^m is ergodic. Since T ergodic implies T^{-1} ergodic, T^m is ergodic for all nonzero integers.

THEOREM. *A necessary and sufficient condition that a metric automorphism T be weak mixing is that it have property* W.

PROOF. Suppose first T is weak mixing. Then (see [2]) for A and B given sets of nonzero measure, there exists a subset K' of integers with density zero such that $\lim_{n\notin K'} P(T^nA\cap B)=P(A)P(B)>0$. Thus for all n not in K' and larger than some integer N, $P(T^nA\cap B)>0$. Let $K=K'\cup\{k: 0\leqq k\leqq N,\ k \text{ integer}\}$. The set K has density zero and if $n\notin K$ then $P(T^nA\cap B)>0$.

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Next suppose T has property W. Then T is ergodic and hence the induced unitary operator on $L_2(P)$ defined by $(U_T f)(x) = f(Tx)$ has simple eigenvalues and eigenfunctions with constant absolute values (see [2]). Suppose T is not weak mixing. Then U_T has an eigenvalue $\lambda \neq 1$ (see [2]). Let f_0 denote the eigenfunction associated with λ such that $|f_0| = 1$. Since λ is not 1 and T is ergodic, f_0 is nonconstant. Also since every power of T is ergodic, λ cannot be a root of unity.

Let S denote the circle and R_λ the λ-rotation on S, i.e. $R_\lambda z = \lambda z$ for all $z \in S$. Since $|f_0| = 1$, the range of f_0, which we shall denote by Y, is a subset of S and by direct computation one may show that Y is invariant under R_λ. If T_λ denotes the restriction of R_λ to Y, then T_λ is a one to one map of Y onto Y.

Let $\mathcal{B}$ denote the Borel subsets of S and $\mathcal{C}$ denote the σ-algebra $\mathcal{B} \cap Y = \{B \cap Y : B \in \mathcal{B}\}$ of subsets of Y. Since $f_0^{-1}(B \cap Y) = f_0^{-1}(B)$ and f_0 is a measurable function, $f_0^{-1}(\mathcal{C}) \subset \mathcal{A}$. Thus f_0 is a measurable function on $(X, \mathcal{A})$ onto $(Y, \mathcal{C})$. Likewise since $T_\lambda^{-1}(B \cap Y) = R_\lambda^{-1}(B) \cap Y$ and R_λ is a Borel map, T_λ is a measurable function on $(Y, \mathcal{C})$ onto $(Y, \mathcal{C})$.

Define the probability P_* on $(Y, \mathcal{C})$ by $P_*(A \cap Y) = Pf_0^{-1}(A)$. If $A \cap Y \in \mathcal{C}$, then

$$P_*[T_\lambda^{-1}(A \cap Y)] = Pf_0^{-1}[R_\lambda^{-1}(\mathrm{A}) \cap Y] = Pf_0^{-1}[R_\lambda^{-1}(A)].$$

Since T is measure preserving $P_*(A \cap Y) = P(T^{-1}(f_0^{-1}A))$. However, $f_0^{-1}(R_\lambda^{-1}A) = T^{-1}(f_0^{-1}A)$ so that $P_*(A \cap Y) = P_*[T_\lambda^{-1}(A \cap Y)]$ and T_λ is P_*-measure preserving. Thus $(Y, \mathcal{C}, P_*)$ is a probability space and T_λ is a metric automorphism of the space. Moreover f_0 is a metric homomorphism of $(X, \mathcal{A}, P)$ onto $(Y, \mathcal{C}, P_*)$ such that $f_0 T = T_\lambda f_0$. That is, the dynamical system $(X, \mathcal{A}, P, T)$ is homomorphic to the dynamical system $(Y, \mathcal{C}, P_*, T_\lambda)$.

Now we show that since T satisfies property W and T_λ is a homomorphic image of T, T_λ must satisfy property W. Let A and B be two sets from $\mathcal{C}$ which have positive P_*-measure and denote $f_0^{-1}(A)$ and $f_0^{-1}(B)$ respectively by A' and B'. Since these two sets have positive P-measure and T has property W there is the set K of density zero associated with A' and B' by property W. Let n be a positive integer not in K. Since $f_0^{-1}(T_\lambda^n A) \cap f_0^{-1}(B) \supset T_\lambda^n(f_0^{-1}(A)) \cap f_0^{-1}(B)$ we have that

$$\begin{aligned} P_*(T_\lambda^n A \cap B) &= P[f_0^{-1}(T_\lambda^n A) \cap f_0^{-1}(B)] \\ &\geqq P(T^n A' \cap B') > 0 \end{aligned}$$

and T_λ satisfies property W.

However, since T_λ is a certain restriction of a rotation on the circle we can show that it cannot satisfy property W and this contradiction will complete the proof.

Suppose there exists a nondegenerate arc I_0 in S such that $P_*(I_0 \cap Y) = 0$. Since λ is not a root of unity, R_λ is an irrational rotation on the circle and hence $\bigcup_{n=0}^{\infty} R_\lambda^n I_0 = S$. Thus

$$1 = P_*(Y) = P_*\left[\bigcup_{n=0}^{\infty} T_\lambda^n(I_0 \cap Y)\right] \leqq \sum_{n=0}^{\infty} P_*(T_\lambda^n(I_0 \cap Y)) = 0$$

and this contradiction implies that for every nondegenerate arc I of S, $P_*(I \cap Y) > 0$.

Now, let I_0 denote the arc of S between 1 and i and I_1 denote the arc of S between -1 and $-i$. Since an irrational rotation is uniformly almost periodic there exists a relatively dense set $\{n_i\}$ of integers such that $R_\lambda^{n_i} I_0 \cap I_0 \neq \emptyset$. But this implies that $R_\lambda^{n_i} I_0 \cap I_1 = \emptyset$ and it follows that $T_\lambda^{n_i}(I_0 \cap Y) \cap (I_1 \cap Y) = \emptyset$ for all i. Since $I_0 \cap Y$ and $I_1 \cap Y$ have positive P_*-measure if T_λ satisfied condition W, then the set $\{n_i\}$ would have to be a subset of a set of density zero and hence itself have density zero. This is impossible since $\{n_i\}$ is relatively dense.

It might be of some interest to point out that the theorem together with the lemma shows that weak mixing automorphisms are strongly ergodic. Also in the proof of the theorem a homomorphic image of a nonweak mixing automorphism was constructed which was a rotation on an invariant subset of the circle. Since this implies that nonweak mixing automorphisms have factor automorphisms with zero entropy we have that an automorphism with completely positive entropy must be weak mixing. However, it is known [3] that automorphisms with completely positive entropy are mixing of all orders.

Finally, we conjecture that a necessary and sufficient condition that a metric automorphism T be strongly mixing is that for every two sets A and B of nonzero measure there exists an integer N such that for all $n \geqq N$, $P(T^n A \cap B) > 0$.

Bibliography

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