This derivation of Theorem 2 from Theorem 1 was shown to us by C. T. C. Wall.

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ON THE NORM OF STABLE MEASURES[1]

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1. Limits of convolution powers and stable measures. Let $M(R)$ denote the Banach algebra of all complex-valued regular finite measures defined on the Borel sets of the real line R, where multiplication is defined by convolution, and

$$\|\mu\| = \sup \sum |\mu(R_i)|,$$

the supremum being taken over all finite collections of pairwise disjoint sets R_i whose union is R. Let $B(R)$ be the set of all Fourier transforms of measures in $M(R)$.

In [1], we characterized all possible limits

$$\lim_{n\to\infty} (\hat{\nu}(t/B_n))^n \exp(itA_n) = \hat{\mu}(t) \quad \text{for all } t \neq 0,$$

where $A_n \in R$, $B_n > 0$, $\hat{\nu}$, $\hat{\mu} \in B(R)$. This is a generalization of an old problem in probability theory (see e.g. [4]). One can show that a measure μ appears as a limit if and only if it is *stable*, i.e. has the following property: For all $a>0$, $b>0$ there exist $c>0$ and $\gamma \in R$ such that

$$(1) \qquad \hat{\mu}(at)\hat{\mu}(bt) = \hat{\mu}(ct)\exp(i\gamma t) \quad \text{for all } t \in R.$$

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In other words, a stable measure convolved with itself reproduces itself after being properly shifted and scaled. Consequently, stable measures may be considered as a substitute for idempotent measures, which except for degenerate ones do not exist on the real line.

Besides the degenerate measure $\mu=0$ and $\mu=\delta_\beta$ (unit mass at $x=\beta$), a measure is a solution of (1) if and only if its Fourier transform is of the form

$$\begin{aligned}\hat{\mu}(t) &= \exp(-c|t|^\alpha + i\beta t) \quad \text{for } t \geqq 0,\\ &= \exp(-d|t|^\alpha + i\beta t) \quad \text{for } t < 0,\end{aligned}$$

or

$$\begin{aligned}\hat{\mu}(t) &= \exp(-c|t| + i\beta t \log |t|) \quad \text{for } t \geqq 0,\\ &= \exp(-d|t| + i\beta t \log |t|) \quad \text{for } t < 0,\end{aligned}$$

where $\beta \in R, \alpha \in R, \alpha \neq 0$; c and d are complex constants with $\operatorname{Re}(c) > 0$, $\operatorname{Re}(d) > 0$. For $\alpha > 0$ the corresponding measure μ is absolutely continuous; for $\alpha < 0$ the measure $\delta_\beta - \mu$ is absolutely continuous.

2. **Symmetric real-valued stable measures.** By (1),

$$\|\mu * \mu\| = \|\mu\| \tag{2}$$

for every stable measure. Therefore, either $\|\mu\|=0$ or $\|\mu\| \geqq 1$. The stable probability measures clearly have $\|\mu\|=1$. One sees easily that $\|\mu\|>1$ for every stable measure which is not a probability measure. For $\alpha<0$ we even have $\|\mu\|>2$.

In this section we confine ourselves to

$$\hat{\mu}_\alpha(t) = \exp(-|t|^\alpha), \qquad \alpha \in R, \alpha \neq 0.$$

Clearly $\|\mu_\alpha\|$ is equal to 1 for $0<\alpha\leqq 2$, is bigger than 1 for $\alpha>2$, and is bigger than 2 for $\alpha<0$.

Our tool will be an approximation of $\hat{\mu}_\alpha$ by a function whose norm can be calculated, and the repeated use of

LEMMA 1 (BEURLING [2]). (i) *Let ϕ be absolutely continuous and ϕ, $\phi' \in L_2(R)$. Then $\phi=\hat{\mu} \in B(R)$, μ is absolutely continuous, and*

$$\|\mu\| \leqq \left(\int_{-\infty}^{\infty} |\phi(t)|^2 dt \int_{-\infty}^{\infty} |\phi'(t)|^2 dt\right)^{1/4}.$$

(ii) *An even function ϕ is in $B(R)$ if $\phi(t) \to 0$ $(t \to \infty)$ and if the integral below is convergent. Then, putting $\phi=\hat{\mu}$,*

$$\|\mu\| \leqq \int_0^\infty t\,|d\phi'(t)|.$$

THEOREM 1. (i) *For* $\alpha<0$,

$$2 < \|\mu_\alpha\| \leqq 2 + ((2\alpha(\alpha-1))^{1/2} - \alpha) \exp(1/\alpha - 1).$$

(ii) *If* $\alpha\beta>0$ *then*

$$\|\mu_\alpha - \mu_\beta\| \leqq |\beta-\alpha| K(\alpha,\beta),$$

where K is locally bounded.

PROOF. (i) The inflection points of $\hat{\mu}_\alpha$ are $\pm t_0$ where $t_0 = ((\alpha-1)/\alpha)^{1/\alpha}$. Approximate $1-\hat{\mu}_\alpha$ by

$$\begin{aligned} g_\alpha(t) &= 1 - \hat{\mu}_\alpha(t) \quad \text{for } |t| > t_0, \\ &= 1 + \hat{\mu}_\alpha'(-t_0)(t+t_0) - \hat{\mu}_\alpha(-t_0) \quad \text{for } -t_0 \leqq t < 0, \\ &= 1 + \hat{\mu}_\alpha'(t_0)(t-t_0) - \hat{\mu}_\alpha(t_0) \quad \text{for } 0 \leqq t \leqq t_0. \end{aligned}$$

The function g_α is even and concave in $(0, \infty)$, and therefore by Polya's criterion is positive definite. Thus $g_\alpha = \hat{\nu}_\alpha$ where $\|\nu_\alpha\| = g_\alpha(0) = 1 - \exp(1/\alpha - 1)$. For the remainder $\hat{\nu}_\alpha - (1-\hat{\mu}_\alpha)$, we find by Lemma 1 (i)

$$\|\nu_\alpha - (\delta_0 - \mu_\alpha)\| \leqq (2\alpha(\alpha-1))^{1/2} \exp(1/\alpha - 1).$$

(ii) Lemma 1(ii) yields

$$\|\hat{\mu}_\alpha - \hat{\mu}_\beta\| \leqq \int_0^\infty t\,|\hat{\mu}_\beta''(t) - \hat{\mu}_\alpha''(t)|\,dt.$$

By the mean value theorem applied to the variable α,

$$\hat{\mu}_\beta''(t) - \hat{\mu}_\alpha''(t) = (\beta-\alpha) \left.\frac{\partial \hat{\mu}''_\gamma(t)}{\partial\gamma}\right|_{\gamma=\alpha+(\beta-\alpha)\theta},$$

$0<\theta<1$. An elementary calculation yields

$$\int_0^\infty t \left| \frac{\partial \hat{\mu}''_\gamma(t)}{\partial \gamma} \right| dt \leqq K(\alpha,\beta),$$

where K is locally bounded.

COROLLARY 1. (i) $\lim_{\alpha\to-0} \|\mu_\alpha\| = 2$.

(ii) *The function* $\alpha\to\mu_\alpha$ *mapping* $R-\{0\}$ *into* $M(R)$ *is continuous with respect to the norm topology in* $M(R)$.

It should be mentioned that if we define $\mu_0 = e^{-1}\delta_0$, then μ_α is continuous at $\alpha=0$ in the weak* topology of $M(R)$.

3. Asymptotic behavior of $\|\mu_\alpha\|$.

THEOREM 2. *As* $|\alpha|\to\infty$,

$$\|\mu_\alpha\| = (4/\pi^2)\log|\alpha| + O(1).$$

For the proof of this fact we need the following

LEMMA 2. *Consider the trapezoid-shaped function*

$$\begin{aligned} \hat{\nu}_{a,b}(t) &= 0 && \text{for } |t| \geqq b, \\ &= (1/a)(b-|t|) && \text{for } b-a \leqq |t| < b, \\ &= 1 && \text{for } |t| < b-a, \end{aligned}$$

where $b>a>0$. *Then, for* $b/a\to\infty$,

$$\|\nu_{a,b}\| = (4/\pi^2)\log(b/a) + O(1).$$

For the proof write $\hat{\nu}_{a,b}=\hat{\sigma}_1+\hat{\sigma}_2$, where

$$\hat{\sigma}_1(t) = \sum_{|k|\leq[b/a]-1} \hat{p}(t+ka)$$

and

$$\begin{aligned} \hat{p}(t) &= 1-|t|/a && \text{for } |t|\leqq a, \\ &= 0 && \text{for } |t|>a. \end{aligned}$$

Lemma 1 (i) applied to $\sigma_2=\nu_{a,b}-\sigma_1$ yields $\|\sigma_2\|\leqq 2$. Furthermore, by direct calculation using Poisson's summation formula, we obtain

$$\|\sigma_1\| = (1/\pi)\int_{-\pi}^{\pi} |D_{[b/a]-1}(x)|\,dx,$$

where D_n is the Dirichlet kernel.

PROOF OF THEOREM 2. Assume $\alpha\geqq 1$. Approximate $\hat{\mu}_\alpha$ by a trapezoid $\hat{\nu}_\alpha$ so that its sides coincide with the tangents at the inflection points of $\hat{\mu}_\alpha$. This leads to $b/a=\alpha\exp(1/\alpha-1)$, and by the above lemma for $\alpha\to\infty$,

$$\|\nu_\alpha\| = (4/\pi^2)\log\alpha + O(1).$$

Again by Lemma 1 (i), $|\,\|\nu_\alpha\|-\|\mu_\alpha\|\,|=O(1)$. We proceed similarly for $\alpha>0$.

In the same way we can show that $\|\mu_\alpha+\mu_{-\alpha}\|=O(1)$. Although $\|\mu_\alpha\|\to\infty$ we have for the densities g_α of μ_α

$$g_\alpha(x)\to(\sin x)/\pi x \quad \text{as } \alpha\to\infty$$

in the norm of $L_2(R)$, as Parseval's equation shows.

COROLLARY 2. *For every* $\epsilon>0$ *there is a* $\mu\in M(R)$ *such that* $\|\mu\|=1$, *but* $\|\mu*\mu\|<\epsilon$.

To see this, choose μ_α such that $\|\mu_\alpha\|>\epsilon^{-1}$. Now take $\mu=\mu_\alpha/\|\mu_\alpha\|$ and use (2).

Corollary 2 is true also in $M(G)$, where G is the circle group or any compact connected abelian group, since in such a group there exist idempotent measures with arbitrarily large norm. See Cohen [3].

4. **A skew case.** Consider the stable measures $\mu_{c,\alpha}$ corresponding to

$$\begin{aligned}\hat{\mu}_{c,\alpha}(t) &= \exp(-c|t|^\alpha) \quad \text{for } t \geqq 0,\\ &= \exp(-|t|^\alpha) \quad \text{for } t < 0,\end{aligned}$$

where $\alpha\notin(0, 1)$ and $c\in R$.

THEOREM 3. *For* $c\to\infty$,

$$2\log c+O(1) \leqq \|\mu_{c,\alpha}\| \leqq 2\,|2\alpha\exp(1/\alpha-1)-1|\log c+O(1).$$

A technique similar to the one used in Theorem 2 leads to the conjugate Fejér kernel rather than to the Dirichlet kernel.

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