THE NONORIENTABLE GENUS OF K_n

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1. Introduction. One of the oldest problems in combinatorics is that of determining the chromatic number of each nonorientable 2-manifold. The problem is equivalent to determining the nonorientable genus of each complete graph, and was solved by Ringel [1] during the last decade using rather complicated methods.

A determination of the nonorientable genus of K_n by quite simple combinatorial techniques was presented in [2] if $n \equiv 3$, 4 or 5 (mod 6). This note is concerned with the situation in case $n \equiv 0$, 1 or 2 (mod 6).

The solution in these cases is not as elegant as that obtained for the other residue classes modulo 6. There certain combinatorial properties of the cyclic group Z_{6t+3} led to an extraordinarily unified solution. One might hope that using the group Z_{6t} would produce similar unification here. This is not the case because the two groups have significantly different combinatorial properties. In fact the solution presented here divides each case into two subcases. To be more precise, the nonorientable genus of K_n is determined for $n = 12s + k$, where $k = 0$, 6; 1, 7; 2, 8.

Details are offered for $k = 0$, 7 and 8; the first two because of an interesting comparison which can be made with the companion problem of determining the orientable genus of K_n. In the orientable problem (not yet solved for all n) the case $k = 0$ was particularly difficult, here it is almost trivial; in the orientable problem the case $k = 7$ is very simple, here it is somewhat involved. The case $k = 8$ is presented because Ringel found it particularly troublesome.

2. Definitions and general comments. In order to save space the reader is referred to [2] for the basic definitions and ideas involved.

Of particular importance is $\tilde{\gamma}(K_n)$, the nonorientable genus of the complete n-graph K_n, and

$$\tilde{I}(n) = \{(n-3)(n-4)/6\},^1 \qquad n = 5, 6, 7, \cdots.$$

We show that if $n \neq 7$ then

$$\tilde{\gamma}(K_n) = \tilde{I}(n). \tag{1}$$

[1] $\{a\}$ is the smallest integer not less than a.

In [2] we worked with the graph K_n^2 if $n=6t+5$. Here, in case $n=12t+8$, we need an additional graph which, like K_n^2, is "almost" complete.

Suppose n is even. Add three distinct vertices x, y_1 and y_2 to K_n. Join x to all the vertices of K_n, join y_1 to *half* the vertices of K_n and y_2 to the *other half*. This new graph is designated by L_{n+2}^2. Note that if y_1 and y_2 are identified to yield a vertex y the result is K_{n+2}^2. And now K_{n+2} is obtained by joining x and y by an arc.

If $n \equiv 0$ or $1 \pmod 6$ then $(n-3)(n-4) \equiv 0 \pmod 6$ and a triangular imbedding of K_n is not ruled out by the Euler characteristic formula. In each of these cases, except $n=7$, we exhibit a triangular imbedding of K_n and thus prove (1).

If $n \equiv 2 \pmod 6$ then $(n-3)(n-4) \equiv 2 \pmod 6$ and a triangular imbedding of K_n is impossible. If $n=12s+2$ we proceed as in case $n=6t+5$ of [2] to obtain, first, a triangular imbedding of K_n^2. If $n=12s+8$ we exhibit a triangular imbedding of L_n^2 in a manifold $\tilde{N}$ whose genus is $\tilde{I}(n)-2$. By adding *two* carefully positioned cross caps to $\tilde{N}$ we obtain a manifold $\tilde{M}$ on which it is possible to identify y_1 and y_2 to get a vertex y, and also join x to y by an arc which intersects no other arc on $\tilde{M}$. Thus we obtain an imbedding of K_n in $\tilde{M}$. Since $\gamma(\tilde{M})=\tilde{I}(n)$ we have proved (1).

3. Construction of nonorientable schemas.

EXAMPLE 1. K_n, $n=12s$, using the symbol z in addition to the elements of Z_{12s-1} to name the vertices.

The solution is shown in the current graph of Figure 1.

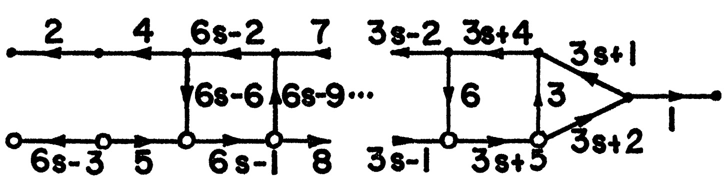

FIGURE 1

Because there are an even number of vertical arcs there will be a single circuit. The cyclic permutation P_0 for the schema is obtained by recording the successive currents on the directed arcs of the circuit and inserting z between 1 and -1. The permutation P_g for g in Z_{12s-1} is obtained from P_0 by the additivity principle of [2]. Finally P_z is $(0, 1, 2, \cdots)$.

It is instructive to consider the situation in detail for $s=1$. Here P_0 is $(4, 2, -2, -4, 1, z, -1, -5, 3, -3, 5)$. Note that $\mathfrak{R}^*$ holds at (g, z)

for g in Z_{11}. (See [2].) Moreover, $\mathcal{R}^*$ holds at $(0, g)$ for $g = \pm 1$, ± 4 and ± 5, while $\mathcal{R}$ holds at $(0, g)$ for $g = \pm 2$ and ± 3. In view of the additivity principle, this implies that $\mathcal{R}^*$ holds at $(h, h+g)$ for the first set of g's above, and that $\mathcal{R}$ holds at $(h, h+g)$ for the second set with any h in Z_{11}. This is a nonorientable schema for K_{12} since the permutation P_z serves to "lock" the array so that condition (ii) of [2] is satisfied.

The properties of Figure 1 which guarantee that the resulting schema is nonorientable are: (i) Kirchhoff's current law holds at each vertex of degree 3. (ii) The current on the arc adjacent to z generates Z_{12s-1}. (iii) If g is a current flowing from a vertex of degree 1 into a vertex of degree 2, and h is the other current entering this last vertex, then $2g+h=0$.

A great deal can be accomplished with the principles involved in this one illustration.

EXAMPLE 2. K_{13}, using the elements of Z_{13} to name the vertices.

The solution is presented in the current graph of Figure 2, and is also to be found in complete schema form in Ringel [1, p. 71] where 13 is to be read as 0.

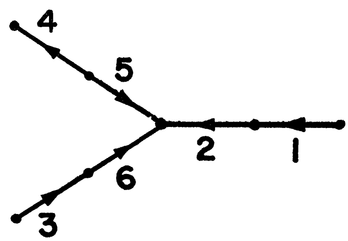

FIGURE 2

The solution for $n = 12s+1$ with $s > 1$ does not follow the pattern of Figure 2 but uses a cascade. In fact no solution using the above current graph as a model has been found.

EXAMPLE 3. K_n, $n = 12s+8$, using the symbols x, y_1 and y_2 in addition to the elements of Z_{12s+6} to name the vertices of L^2_{12s+8}.

A solution is given for $s=2$ in Figure 3.

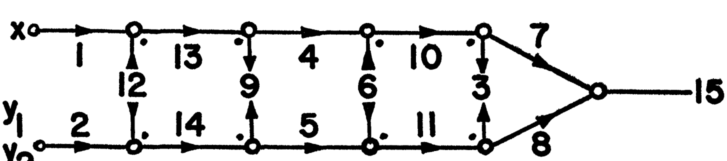

FIGURE 3

This is a further example of a cascade. The cyclic permutation P_0 is obtained in a fashion directly analogous to the development of Example 1 in [2], except that y_2 appears between -2 and 2. As before, P_g is obtained by the additivity principle except that y_2 is inserted between $(-2+g)$ and $(2+g)$ if g is even, and y_1, if g is odd. Finally $P_x=(0,\ -1,\ -2,\cdots)$, $P_{y_1}=(-1,\ -3,\ -5,\cdots)$ and $P_{y_2}=(0,\ -2,\ -4,\cdots)$. A general theorem guarantees that the result is a nonorientable schema for L_{32}^2.

The idea is fairly transparent and there should be no difficulty in getting cascades of the above type for any positive s. If $s=0$ then the schema for L_8^2 is *orientable* and we get a nonorientable imbedding of K_8 only when we add the two cross caps mentioned in §2.

As a matter of fact it is a simple matter to obtain a current graph which will provide an *orientable* schema for L_{12s+8}^2 by using a ladder like diagram with currents 1 and $(6s+2)$ on the extreme left and $(6s+3)$ on the right. If one could now identify y_1 and y_2 on the orientable manifold to get a vertex y and then join y to x by adding *one* handle (instead of two cross caps) the orientable problem would be solved for $n=12s+8$. This does not appear to be possible, and the solution to the orientable case is still open. However, it was because of an attack on the orientable problem that the above solution for the nonorientable case was discovered.

EXAMPLE 4. K_n, $n=12s+7$, using the symbol z in addition to the elements of Z_{12s+6} to name the vertices.

A solution is given for $s=2$.

Example 4 has exactly the same relation to Example 3 that Example 2 of [2] has there to Example 1. There are three leafs, as in Figure 4 of [2], and a vortex -3 containing z. The permutation P_z is $(2, 0, 1;\ -1, -3, -2;\ -4, -6, -5;\cdots)$. A general theorem guarantees that the result is a nonorientable schema for K_{31}.

The comments made at the end of Example 3 show why, in case $s=0$, this device will not provide a nonorientable schema for K_7. In fact, $n=7$ is an exceptional case, and no nonorientable schema can exist.

The solutions to the remaining cases, $n\equiv 1$, 2, and 6 (mod 12) were not without a certain element of drama. All were solved except $n=12s+1$ with *s even* at the beginning of a conference on *Graph theory* at Oberwolfach in July, 1967. Richard K. Guy of Calgary got extremely interested in the problem as a consequence of my lecture on July 1, and by July 4 we had solved this last "half case."

References

1. Gerhard Ringel, *Färbungsprobleme auf Flächen und Graphen*, Deutscher Verlag, Berlin, 1959.
2. J. W. T. Youngs, *Remarks on the Heawood conjecture (non-orientable case)*, Bull. Amer. Math. Soc. 74 (1968), 347–353.

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GLOBAL ASYMPTOTIC ESTIMATES FOR ELLIPTIC SPECTRAL FUNCTIONS AND EIGENVALUES[1]

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The asymptotic behavior of the spectral function of a selfadjoint elliptic operator has been studied extensively; cf. the discussion in [1] and [7]. Recently Agmon and Kannai [2] and Hörmander [8] have obtained error estimates for general operators. Most of this work is concerned with interior estimates for operators with rather smooth coefficients. Here we consider behavior up to the boundary, with minimal assumptions on the coefficients. Details and proofs will appear elsewhere.

Let $\mathcal{A}=\sum a_\alpha(x)D^\alpha$ be an operator of order $m=2r$ defined on a region Ω in R^n. We assume that the boundary $\partial\Omega$ is uniformly regular of class $m+1$ in the sense of [6]. Let $B_j=\sum b_{j,\beta}(x)D^\beta$, $j=1, 2, \cdots, r$, be an operator of order $m_j<m$ defined on an ϵ-neighborhood of $\partial\Omega$. Suppose $0<h\leqq 1$. We assume

(1) $\mathcal{A}$ is uniformly strongly elliptic on Ω.

$(2)_h$ The coefficients a_α are bounded and measurable on Ω. For $|\alpha|=m$ and x, y in Ω,

$$|a_\alpha(y)-a_\alpha(x)| \leqq c|y-x|^h.$$

$(3)_h$ The coefficients $b_{j,\beta}$ and their derivatives of order $\leqq m-m_j$ are bounded and continuous on Ω. For $|\beta|=m_j$ and $|\gamma|=m-m_j$,

$$|D^\gamma b_{j,\beta}(y)-D^\gamma b_{j,\beta}(x)| \leqq c|y-x|^h.$$

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