FLATTENING A SUBMANIFOLD IN CODIMENSIONS ONE AND TWO

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Let M and N be manifolds with $M \subset \text{Int } N$, and assume that $M-X$ is locally flat in N, where X is some subset of M. We are interested in finding conditions (intrinsic, placement, dimensional, etc.) which, when placed on X, imply that M is locally flat in N. Extremely useful and satisfying answers are provided by Bryant and Seebeck in [2], assuming that $\dim N - \dim M \geqq 3$. We announce here a method for deducing local versions of Corollary 1.1 of [2] in codimensions one and two.

DEFINITIONS. If M is a manifold, a *collaring* of Bd M in M is an embedding λ of Bd $M \times [0, \infty)$ into M such that $\lambda(x, 0)=x$ for each x in Bd M. We use R^n to denote euclidean n-space, B^n the closed unit ball in R^n.

THEOREM. *For integers $0 \leqq k < m \leqq n$, let D be an m-cell in R^n and let E be a k-cell in* Bd D. *Assume that the following condition is satisfied*:

$$D - E \textit{ is locally flat in } R^n, \quad \textit{and} \quad E \textit{ is locally flat in } \mathrm{Bd}\, D.$$

Then $(R^n, D) \approx (R^n, B^m)$ if and only if $\lambda(E \times I)$ is locally flat in R^n for some collaring λ of Bd D *in D.*

The proof of this theorem is similar to the proof of Theorem 4.2 of [7]. Theorem 4.1 of [7] must be used more carefully to replace Corollary 3.2 of [7].

A detailed proof of the above theorem, together with applications and generalizations, will appear elsewhere. We present below the immediate implications of [2]. (Actually, in an earlier paper which is in press, Bryant and Seebeck prove a local form of Corollary 1.1 of [2] which is enough to yield the following applications.)

REMARK. There are no dimensional restrictions (other than $0 \leqq k < m \leqq n$) in the above Theorem.

APPLICATION 1. *Let D be an m-cell in R^n, and let E be a k-cell in* Bd D. *Assume that*

$$D - E \textit{ and } E \textit{ are locally flat in } R^n, \quad \textit{and} \quad E \textit{ is locally flat in } \mathrm{Bd}\, D.$$

If $k \leqq n-4$ then $(R^n, D) \approx (R^n, B^m)$.

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PROOF. Let λ be a collaring of Bd D in D. If $n \geqq 4$ and $k=0$, then $\lambda(E \times I)$ is locally flat in R^n by [3]. If $n \geqq 5$, $\lambda(E \times I)$ is locally flat in R^n by Corollary 1.1 of [2]. In either case the result follows from our Theorem.

REMARKS. 1. The analogue of Application 1 for $k=n-3 \geqq 0$ is false; the Theorem may still be applied to specific cases, however.

2. There are no restrictions on m and n in Application 1.

DEFINITION. Let $\beta(n)$ denote the following conjecture: If D_1 and D_2 are locally flat $(n-1)$-cells in R^n such that $D_1 \cap D_2 = \text{Bd } D_1 \cap \text{Bd } D_2$ is an $(n-2)$-cell whose boundary is locally flat in both Bd D_1 and Bd D_2, then $D_1 \cup D_2$ is locally flat in R^n.

Conjecture $\beta(3)$ is proved in [5]. A proof of $\beta(n)$, $n \geqq 5$, is announced and outlined by Černavskiĭ in [4]. $\beta(4)$ has recently been proved by Černavskiĭ and by R. C. Kirby.

APPLICATION 2. *Let D be an $(n-1)$-cell in R^n, and let E be a k-cell in D. Assume that*

(D, E) is a proper locally flat cell pair, and
$D - E$ and E are locally flat in R^n.

If $k \leqq n-4$ then $(R^n, D) \approx (R^n, B^{n-1})$.

PROOF. Let f: $(B^{n-1}, B^k) \approx (D, E)$ be a homeomorphism. (See [6].) Let $D_1 = f(B^{n-1}_+)$ and $D_2 = f(B^{n-1}_-)$. By Application 1, D_1 and D_2 are locally flat. By $\beta(n)$, D is locally flat.

REMARKS. 1. The analogue of Application 2, with D an $(n-2)$-cell, is false for $n \geqq 3$.

2. The Theorem and Applications can be applied locally to embeddings of manifolds.

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