A FIXED POINT THEOREM OF THE ALTERNATIVE, FOR CONTRACTIONS ON A GENERALIZED COMPLETE METRIC SPACE

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1. **Summary.** The purpose of this note is to prove a "theorem of the alternative" for any "contraction mapping" T on a "generalized complete metric space" X. The conclusion of the theorem, speaking in general terms, asserts that: *either* all consecutive pairs of the sequence of successive approximations (starting from an element x_0 of X) are infinitely far apart, *or* the sequence of successive approximations, with initial element x_0, converges to a fixed point of T (what particular fixed point depends, in general, on the initial element x_0). The present theorem contains as special cases both Banach's [1] contraction mapping theorem for complete metric spaces, and Luxemburg's [2] contraction mapping theorem for generalized metric spaces.

2. **A fixed point theorem.** Following Luxemburg [2, p. 541], the concept of a "generalized complete metric space" may be introduced as in this quotation:

"Let X be an abstract (nonempty) set, the elements of which are denoted by $x, y, \cdots$ and assume that on the Cartesian product $X \times X$ a distance function $d(x, y)(0 \leqq d(x, y) \leqq \infty)$ is defined, satisfying the following conditions

(D1) $d(x, y) = 0$ if and only if $x = y$,

(D2) $d(x, y) = d(y, x)$ (symmetry),

(D3) $d(x, y) \leqq d(x, z) + d(z, y)$ (triangle inequality),

(D4) every d-Cauchy sequence in X is d-convergent, i.e. $\lim_{n,m\to\infty} d(x_n, x_m) = 0$ for a sequence $x_n \in X$ $(n = 1, 2, \cdots)$ implies the existence of an element $x \in X$ with $\lim_{n\to\infty} d(x, x_n) = 0$, ($x$ is unique by (D1) and (D3)).

This concept differs from the usual concept of a complete metric space by the fact that not every two points in X have necessarily a finite distance. One might call such a space a generalized complete metric space."

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Using this notion, one has the

THEOREM. *Suppose that (X, d) is a generalized complete metric space, and that the function $T\colon X \to X$ is a "contraction," that is, T satisfies the condition*: (C1) *There exists a constant q, with $0<q<1$, such that whenever $d(x, y)<\infty$ one has*

$$d(Tx, Ty) \leqq qd(x, y).$$

Let $x_0 \in X$, and consider the "sequence of successive approximations with initial element x_0": $x_0, Tx_0, T^2x_0, \cdots, T^lx_0, \cdots$. Then the following alternative holds: either

(A) *for every integer $l=0, 1, 2, \cdots$, one has*

$$d(T^lx_0, T^{l+1}x_0) = \infty, \quad or$$

(B) *the sequence of successive approximations $x_0, Tx_0, T^2x_0, \cdots, T^lx_0, \cdots$, is d-convergent to a fixed point of T.*

PROOF. Consider the sequence of numbers $d(x_0, Tx_0)$, $d(Tx_0, T^2x_0)$, $\cdots$, $d(T^lx_0, T^{l+1}x_0)$, $\cdots$, the "sequence of distances between consecutive neighbors" of the sequence of successive approximations with initial element x_0. There are two mutually exclusive possibilities: *either*

(a) for every integer $l=0, 1, 2, \cdots$, one has

$$d(T^lx_0, T^{l+1}x_0) = \infty,$$

(which is precisely the alternative (A) of the conclusion of the theorem), *or* else

(b) for some integer $l=0, 1, 2, \cdots$, one has

$$d(T^lx_0, T^{l+1}x_0) < \infty.$$

In order to complete the proof it only remains to show that (b) implies alternative (B) of the conclusion of the theorem.

In case (b) holds, let $N=N(x_0)$ denote a particular one (for definiteness, one could choose the smallest) of all the integers $l=0, 1, 2, \cdots$ such that

$$d(T^lx_0, T^{l+1}x_0) < \infty.$$

Then, by (C1), since $d(T^Nx_0, T^{N+1}x_0)<\infty$, it follows that

$$d(T^{N+1}x_0, T^{N+2}x_0) = d(TT^Nx_0, TT^{N+1}x_0) \leqq qd(T^Nx_0, T^{N+1}x_0) < \infty;$$

and, by mathematical induction, that

$$d(T^{N+l}x_0, T^{N+l+1}x_0) \leqq q^ld(T^Nx_0, T^{N+1}x_0) < \infty,$$

for every integer $l=0, 1, 2, \cdots$. In other words, it has just been proved that, if n is any integer such that $n>N$, then

$$d(T^n x_0, T^{n+1}x_0) \leqq q^{n-N} d(T^N x_0, T^{N+1}x_0) < \infty.$$

But now, the triangle inequality (D3) implies that, whenever $n>N$, one has, for any $l=1, 2, \cdots$, that

$$\begin{aligned} d(T^n x_0, T^{n+l}x_0) &\leqq \sum_{i=1}^{l} d(T^{n+i-1}x_0, T^{n+i}x_0) \\ &\leqq \sum_{i=1}^{l} q^{n+i-1-N} d(T^N x_0, T^{N+1}x_0) \\ &\leqq q^{n-N} \cdot \frac{1-q^l}{1-q} \cdot d(T^N x_0, T^{N+1}x_0). \end{aligned}$$

Therefore, since $0<q<1$, the sequence of successive approximations $x_0, Tx_0, T^2x_0, \cdots, T^n x_0, \cdots$, is a d-Cauchy sequence; and, by (D4), it is d-convergent. That is to say, there exists an element x in X such that $\lim_{n\to\infty} d(T^n x_0, x)=0$.

It will now be shown that x is a fixed point of T. For, whenever $n>N$, from (D3) and (C1),

$$\begin{aligned} 0 \leqq d(x, Tx) &\leqq d(x, T^n x_0) + d(T^n x_0, Tx) \\ &\leqq d(x, T^n x_0) + q d(T^{n-1}x_0, x); \end{aligned}$$

and thus, taking $\lim_{n\to\infty}$, it follows that $d(x, Tx)=0$. Using (D1), this gives $x=Tx$, that is, x is a fixed point of T. This completes the proof.

3. **Remarks.** 1. Banach's [1] contraction mapping theorem is a special case of the present theorem. (Banach's theorem asserts that, if T is a contraction on a complete metric space X, *then* T has exactly one fixed point, and the sequence of successive approximations x_0, $Tx_0, T^2x_0, \cdots, T^l x_0, \cdots$, for any x_0 in X, always converges to the unique fixed point of T.) This can be seen as follows: if X is a complete metric space, then $d(x, y)<\infty$ for every x, y in X, and alternative (A) is excluded; since X is not empty, choosing $x_0 \in X$ gives, from alternative (B), the existence of at least one fixed point of T; finally, since T is a contraction, see (C1), T can have at most one fixed point, because if $x=Tx$ and $y=Ty$ then

$$d(x, y) = d(Tx, Ty) \leqq q d(x, y),$$

which means that $d(x, y)=0$, and $x=y$ from (D1).

2. Luxemburg's [2] contraction mapping theorem for generalized metric spaces is also a special case of the present theorem. (Luxem-

burg's theorem asserts that, if T is a contraction, i.e. T satisfies (C1), on a generalized complete metric space X, and T also satisfies the two additional conditions [**2**, p. 541]: "(C2) For every sequence of successive approximations $x_n = Tx_{n-1}$, $n=1, 2, \cdots$, where x_0 is an arbitrary element of X, there exists an index $N(x_0)$ such that $d(x_N, x_{N+1}) < \infty$ for all $l=1, 2, \cdots$. (C3) If x and y are two fix points of T, i.e. $Tx=x$ and $Ty=y$, then $d(x, y) < \infty$," *then* T has exactly one fixed point, and the sequence of successive approximations $x_0, Tx_0, T^2x_0, \cdots, T^1x_0, \cdots$, for any x_0 in X, always converges to the unique fixed point of T.) This can be seen as follows: in view of hypothesis (C2), alternative (A) is excluded; since X is not empty, choosing $x_0 \in X$ gives, from alternative (B), the existence of at least one fixed point of T; finally, (C3) implies that T can have at most one fixed point, because if $x = Tx$ and $y = Ty$, with $x \neq y$, then (C3) gives that $d(x, y) < \infty$, while (C1) then yields

$$d(x, y) = d(Tx, Ty) \leqq qd(x, y),$$

which means that $d(x, y) = 0$, and $x = y$ from (D1), contradicting the initial $x \neq y$. (This very last bit of reasoning just amounts to saying that if T satisfies (C1), then every two of its fixed points must be infinitely far apart; incidentally, this shows that the situation illustrated in the example in Remark 3 on page 542 of [**2**] is, in a sense, the rule rather than the exception.)

3. The fixed point theorem of §2 applies to a "global" contraction T. When T is only "locally" a contraction (see Luxemburg [**3**, p. 94, condition (C1)]), slight modifications of the proof of §2 yield the following "local" theorem of the alternative, which includes Luxemburg's local theorem [**3**, p. 95] as a special case.

THEOREM. *Suppose that (X, d) is a generalized complete metric space, and that the function $T: X \to X$ is locally a contraction, that is, T satisfies the condition*

(C1)′ *There exists a constant q, with $0 < q < 1$, and a positive constant C, such that whenever $d(x, y) \leqq C$ one has*

$$d(Tx, Ty) \leqq qd(x, y).$$

Let $x_0 \in X$, and consider the sequence of successive approximations with initial element x_0: $x_0, Tx_0, T^2x_0, \cdots, T^lx_0, \cdots$. Then the following alternative holds: either

(A) *for every integer $l = 0, 1, 2, \cdots$, one has $d(T^lx_0, T^{l+1}x_0) > C$, or*

(B) *the sequence of successive approximations $x_0, Tx_0, T^2x_0, \cdots, T^lx_0, \cdots$ is d-convergent to a fixed point of T.*

4. In all the fixed point theorems under discussion, the essential idea is the proof of the convergence of a sequence of successive approximations by means of the geometric series $1+q+q^2+\cdots$. This basic idea, of employing this geometric series as a comparison series, goes back to Banach, and to Picard $\cdots$.

References

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