ON A THEOREM OF NIKODYM WITH APPLICATIONS TO WEAK CONVERGENCE AND VON NEUMANN ALGEBRAS

BY R. B. DARST

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Nikodym's theorem [2, Theorem 4, pp. 309–311] is a "striking improvement of the principle of uniform boundedness" in the space of countably additive measures on a sigma algebra. It says that if a set T of countably additive measures μ on a sigma algebra S is pointwise bounded ($\sup_{\mu\in T} |\mu(E) < \infty$, $E\in S$), then it is uniformly bounded ($\sup_{\mu\in T} (\sup_{E\in S} |\mu(E)|) < \infty$).

Notice that the content of Nikodym's theorem is not changed by assuming that T is a countable set, and recall that a countably additive, complex-valued measure on a sigma algebra is bounded and finitely additive.

The following elementary example illustrates that the theorem cannot be extended to the case of bounded and countably additive measures on an algebra of sets. Suppose that X is the set of nonnegative integers and that a subset E of X is in S if either E or $X-E$ is a finite subset of the positive integers. Let $\{\mu_n\}$ be defined on S by $\mu_n(E)=n$ if E is a finite set containing n, $\mu_n(E)=0$ if E is a finite set not containing n, and $\mu_n(E)=-\mu_n(X-E)$, for every E in S. Then the sequence $\{\mu_n\}$ of bounded and countably additive measures on S is pointwise convergent to zero but not uniformly bounded. Indeed, $\lim_n (\sup_{E\in S} \mu_n(E)) = \infty$.

Nevertheless, via a "sliding hump" argument, Nikodym's theorem can be established for bounded and finitely additive measures on a sigma algebra.

THEOREM. *If $\{\mu_n\}$ is a sequence of bounded and finitely additive measures on a sigma algebra S of subsets of a set X such that, for each element E of S,* $\sup_n |\mu_n(E)| < \infty$, *then* $\sup_n(\sup_{E\in S}|\mu_n(E)|) < \infty$.

Bounded linear functionals on a von Neumann algebra can be identified with bounded and finitely additive measures on a sigma algebra. Because the measures identified with normal functionals are countably additive, J. Aarnes [1] was able to obtain results about normal functionals by applying Nikodym's theorem. The present version of Nikodym's theorem can be employed to obtain the appropriate analogues of Aarnes results, without hypothesizing normality.

THEOREM 1. *If a family $\mathfrak{F}$ of functionals on a von Neumann algebra $\mathfrak{A}$ is pointwise bounded on the projections in $\mathfrak{A}$, then $\mathfrak{F}$ is uniformly bounded on bounded sets of $\mathfrak{A}$.*

THEOREM 2. *Let $\{\phi_n\}_{n\in N}$ be a sequence of linear functionals on $\mathfrak{A}$, and suppose that for every projection $P\in\mathfrak{A}$, $\phi(P)=\lim_n \phi_n(P)$ exists as a finite complex number. Then*

(i) *ϕ has a unique extension to all of $\mathfrak{A}$ as an element of $\mathfrak{A}^*$, and $\lim \phi_n(A)$ exists and is equal to $\phi(A)$ for every $A\in\mathfrak{A}$.*

(ii) *If $\mathcal{P}$ is the set of projections in $\mathfrak{A}$ and $\mathfrak{B}$ is any commutative von Neumann subalgebra of $\mathfrak{A}$, then the family of restrictions $\{\phi_n | \mathcal{P}\cap\mathfrak{B}\}_{n\in N}$ is equicontinuous in 0 with respect to the relativized weak operator topology on $\mathfrak{B}$.*

REFERENCES

1. J. Aarnes, *The Vitali-Hahn-Saks theorem for von Neumann algebras*, Math. Scand. **18** (1966), 87–92.
2. N. Dunford and J. T. Schwartz, *Linear operators*. Vol. I, Interscience, New York, 1958.

PURDUE UNIVERSITY