REPRESENTATION OF F-RINGS

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Consider a lattice ordered algebra A with identity over the rationals $\boldsymbol{Q}$; A is called an *f-ring* if $a \wedge b = 0$, $c \geqq 0$, implies that $ca \wedge b = ac \wedge b = 0$. The maximal l-ideals $\mathfrak{M}$ of A form a compact Hausdorff space in the hull-kernel topology. If A is archimedean, i.e. a so called Φ-algebra, then it is known [5] that A is isomorphic to a subalgebra of the partial algebra $D(\mathfrak{M})$ of all continuous functions f: $\mathfrak{M} \rightarrow \boldsymbol{R} \cup \{\pm \infty\}$ which are finite on a dense open set. The fact that there is a sizable theory of Φ-algebras [5], [6], [10] with no counter part for the more general class of f-rings may partly be due to the existence of this representation, $A \subseteq D(\mathfrak{M})$ for Φ-algebras, and a lack of such a representation in the nonarchimedean case. This latter representation has the defect that it is not onto. Even when $D(\mathfrak{M})$ is an algebra, A need not be all of $D(\mathfrak{M})$. Our objective is to give a representation which not only corrects this defect, but also is applicable to a wider class of f-rings. This new representation will show that the "f" in the term "f-ring" is well justified.

Define $E = \bigcup \{A/M \mid M \in \mathfrak{M}\}$; π: $E \rightarrow \mathfrak{M}$, $\pi^{-1}(M) = A/M$. Each $a \in A$ gives a map $\hat{a}$: $\mathfrak{M} \rightarrow E$, $\hat{a}(M) = a + M$. For any subset $A_1 \subseteq A$, set $\hat{A}_1 = \{\hat{a} \mid a \in A_1\}$. In order that $A \cong \hat{A}$, the condition (A) will be assumed throughout to hold

(A) $\bigcap \mathfrak{M} = \{0\}$.

Appropriate topologies can be introduced in E and $\mathfrak{M}$ making π into a structure which generalizes sheaves and fiber bundles—a so called *field*. (For a complete theory of fields, see [3].) The topologies on E and $\mathfrak{M}$ are unique in a certain well-defined sense. Let $\Gamma(\mathfrak{M}, E)$ be the l-group of all continuous cross sections σ: $\mathfrak{M} \rightarrow E$ with $\pi \circ \sigma$ the identity on $\mathfrak{M}$. Then π is continuous and $\hat{A} \subseteq \Gamma(\mathfrak{M}, E)$ is an l-subgroup. Let A^* be the subalgebra $A^* \equiv \{a \in A \mid |a| < r1, \text{some } 0 < r \in \boldsymbol{Q}\}$. Then $\Gamma(\mathfrak{M}, E)^* \equiv \{\sigma \in \Gamma(\mathfrak{M}, E) \mid |\sigma| < \hat{a} \text{ for some } a \in A^*\}$ is a convex l-subgroup of $\Gamma(\mathfrak{M}, E)$.

Although for ease of exposition, A here is the additive group of a ring, the multiplicative structure of A has not been used thus far. The above construction will be carried out more generally for an arbitrary l-group A and any set of prime subgroups $\mathfrak{M}$ with $\bigcap \mathfrak{M} = \{0\}$.

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If M is not normal in A, then A/M is not a group but merely a right coset space.

Returning now to our previous assumptions, the algebra A is an additive topological group with $\{a\in A \mid |a| < r1\}$, $0 < r\in Q$, as zero neighborhoods. It is possibly non-Hausdorff. If A is complete in this uniform structure, it will be said to be *uniformly closed.* Under obvious pointwise operations, $\Gamma(\mathfrak{M}, E)$ is an l-group; $\hat{A}$ is said to be *uniformly dense* in $\Gamma(\mathfrak{M}, E)$ if for any $\sigma\in\Gamma(\mathfrak{M}, E)$ and any $0<r\in Q$, there is an $a\in A$ with $|\hat{a}-\sigma| < r\hat{1}$. Since we are interested in cases when $\hat{A}=\Gamma(\mathfrak{M}, E)$, or when at least $\hat{A}$ is uniformly dense in $\Gamma(\mathfrak{M}, E)$, besides the assumption (A) various of the following hypotheses will have to be imposed:

(B) A^* is closed under *bounded inversion,* i.e. if $1 < a\in A^*$, then $1/a\in A$.

(B′) A is closed under bounded inversion.

(C) A is uniformly closed.

Since $A^*\subseteq C(\mathfrak{M})$, the ring of real continuous functions, (A) and (C) imply (B).

For representation purposes it is important that the algebra has the property described in the next definition.

1. DEFINITION. A subset A_1 of A contains *positive bounded partitions* of *identity* on $\mathfrak{M}$, if for any open cover $\mathfrak{M}=U_1\cup \cdots \cup U_n$, there are $e_j\in A_1$ satisfying $e_j\in\cap\mathfrak{M}\backslash U_j$; $0\leqq e_j\leqq 1$ for $j=1, \cdots, n$; and $1=e_1+ \cdots +e_n$.

The proof of the next lemma is obtained by using the hull-kernel topology together with the lattice properties of A.

2. LEMMA. *If conditions* (A) *and* (B) *hold, then A^* contains positive bounded partitions of identity on $\mathfrak{M}$.*

It should be noted that A may be noncommutative even if A^* is abelian.

3. LEMMA. *If* (A) *and* (B) *hold, then the following conditions are all equivalent*:

(i) *A^* is archimedean*;

(ii) *A is Hausdorff*;

(iii) *E is Hausdorff.*

The next proposition is not only needed to identify the fibers, but it is also of independent interest.

4. PROPOSITION. *Consider any totally ordered f-ring A such that every $1 < a\in A$ has a two sided inverse in A. Let I be the set of invertible ele-*

ments and define N *as the set* $N=\{x\in A \mid |x|<|i| \text{ all } i\in I\}$. *Then*:

(i) N *is a maximal ideal of* A *which is an l-ideal.*

(ii) A/N *is a totally ordered division ring.*

Very easily describable necessary and sufficient conditions for embedding a rational f-algebra into a real f-algebra do not seem to be available (see [7, p. 351, 2.9] and [11]).

5. COROLLARY. *If the f-ring* A *satisfies conditions* (A) *and* (B′), *then each* A/M *is a totally ordered division ring. Furthermore,* A *can be embedded in an f-algebra over the reals.*

The previous lemmas are now used to obtain the main theorem.

6. THEOREM. *Suppose* A *is an f-algebra with identity over the rationals* Q. *Define* $A^*=\{a\in A \mid |a|<r1 \text{ for some } r\in Q\}$ *and* $\mathfrak{M}$ *as the set of all maximal l-ideals of* A. *Assume that*

(A) $\bigcap\mathfrak{M}=\{0\}$;

(B) $1<a\in A^*\Rightarrow 1/a\in A^*$.

Let π: $E=\bigcup\{A/M \mid M\in\mathfrak{M}\}\to\mathfrak{M}$, $\hat{A}^*$, $\hat{A}$, $\Gamma(\mathfrak{M}, E)$, *and* $\Gamma(\mathfrak{M}, E)^*$ *be as in the introduction.*

(i) *There is a field* π *where* $\mathfrak{M}$ *has the hull-kernel topology. Each* A/M, $M\in\mathfrak{M}$, *is a totally ordered integral domain. There are l-isomorphisms*

$$A \to A \subseteq \Gamma(\mathfrak{M}, E), \qquad A^* \to A^* \subseteq \Gamma(\mathfrak{M}, E)^*.$$

(ii) $\hat{A}$ *is uniformly dense in* $\Gamma(\mathfrak{M}, E)$.

Now assume conditions (A) *and* (C), *where*

(C) A *is uniformly closed.*

Then the following two assertions are valid:

(iii) $A^*\cong\hat{A}^*=C(\mathfrak{M})\hat{1}=\Gamma(\mathfrak{M}, E)^*$; E *is Hausdorff.*

(iv) $\hat{A}=\Gamma(\mathfrak{M}, E)$.

By using Proposition 4 and imposing more hypotheses, we can obtain additional information in the above Theorem.

7. COROLLARY. *With the same notation as in the previous theorem, assume* (A) *and* (B′):

(B′) $1<a\in A\Rightarrow 1/a\in A$.

Then conclusions (i) *and* (ii) *of the previous theorem hold. Furthermore, each* $\pi^{-1}(M)$, $M\in\mathfrak{M}$, *is a totally ordered division ring.*

A converse theorem can also be formulated. One starts from a field π: $E\to\mathfrak{M}$ over a compact Hausdorff space whose stalks are totally ordered integral domains. Then an appropriate subalgebra Λ, in

$\Lambda \subseteq \Gamma(\mathfrak{M}, E)$ is shown to be an f-algebra satisfying the algebraic hypotheses (A) and (B) of the previous theorem.

The full proofs of these results will appear elsewhere later.

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