ISOMETRIES OF L^p-SPACES ASSOCIATED WITH FINITE VON NEUMANN ALGEBRAS

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1. **Introduction.** The object of this paper is to study the isometries of the L^p-spaces, $1 \leqq p < \infty$, associated with a faithful normal semifinite trace on a von Neumann algebra M, and their connections with *-automorphisms of M (see [2], [8] for L^p-spaces, [3] for von Neumann algebras). As is well known, every *-automorphism (or *-anti-automorphism) of a finite factor M induces an L^2-isometry on M. The problem we consider is the converse: under what conditions does an L^p-isometry induce a *-automorphism? Our purpose is to provide a method for constructing *-automorphisms of von Neumann algebras.

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2. **Preliminaries.** Let M be a von Neumann algebra with a faithful normal semifinite trace ϕ. Let m_ϕ be the ideal of trace operators relative to ϕ (see [3, p. 80]). If $0 < \alpha < +\infty$, m_ϕ^α denotes the ideal in M whose positive elements are the operators x^α for x a positive operator in m_ϕ. We have $m_\phi^\alpha \subset m_\phi^\beta$ if $\alpha \geqq \beta > 0$. If ϕ is finite then $M = m_\phi = m_\phi^1$ [2, p. 10]. For $1 \leqq p < \infty$ the set $m_\phi^{1/p}$ equipped with the norm $\|x\|_p = \phi(|x|^p)^{1/p}(|x| = (x^*x)^{1/2})$ is a complex normed linear space, whose completion is called the L^p-space associated with ϕ and M (see [2, pp. 23–27]). We denote this space by $L^p(\phi)$. $L^\infty(\phi)$ denotes the space M with the operator norm. It is known that $L^\infty(\phi)$ is the Banach space dual of $L^1(\phi)$ [3, p. 105], and that $L^p(\phi)$ is the Banach space dual of $L^q(\phi)$ where $1 < p < \infty$ and $1/p + 1/q = 1$, [2, p. 27]. We use the symbol $\langle\, ,\, \rangle$ to denote these dualities and remark that if $x \in m_\phi^{1/p}$ and $y \in m_\phi^{1/q}$, then $\langle x, y \rangle = \phi(xy)$ (here, if $p = 1$, $m_\phi^{1/q}$ denotes the strong closure of m_ϕ) [2, p. 27]. The space $m_\phi^{1/2}$, with the inner product $(x|y) = \phi(y^*x)$, is a pre-Hilbert space whose completion is none other than $L^2(\phi)$.

If M acts on a Hilbert space H, a closed dense linear transformation z in H is affiliated with M if $uzu^{-1} = z$ for all unitary operators u in the commutant of M (see remark following Theorem 1).

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3. **The isometries.** The isometries of $L^\infty(\phi)$ have been completely determined in [5]. The result, which will be used below, is that a linear operator norm isometry ($=L^\infty$-isometry) T of any von Neumann algebra M onto M has the form $x \rightarrow u\rho(x)$ where u is a unitary operator in M and ρ is a C^*-automorphism (=Jordan *-automorphism) of M, that is, $\rho(x^2)=\rho(x)^2$ and $\rho(x)^*=\rho(x^*)$, [5, Theorem 7]. Each C^*-automorphism ρ of M is the direct sum of a *-isomorphism and a *-anti-isomorphism in the following sense: there is a central projection e in M such that $x \rightarrow \rho(x)e$ is a *-isomorphism and $x \rightarrow \rho(x)(1-e)$ is a *-anti-isomorphism, [5, Theorem 10]. Thus each C^*-automorphism of a factor is either a *-automorphism or a *-anti-automorphism.

THEOREM 1. *Let M be a von Neumann algebra with a faithful finite normal trace ϕ, and let T be a linear isometry of $L^1(\phi)$ onto $L^1(\phi)$. Then there is a C^*-automorphism α of M, a positive operator $z \in L^2(\phi)$ affiliated with the center Z of M, and a unitary operator u in M such that*

$$T(x) = \alpha(x)z^2u, \qquad x \in M.$$

REMARK. Since z may be unbounded, all products or sums of operators involving z are "strong products" and "strong sums," as defined in [8, p. 414].

COROLLARY. *If in Theorem 1, M is a factor and $T(I)=I$, then T (restricted to M) is a *-automorphism or a *-anti-automorphism of M.*

PROOF OF THEOREM 1. The Banach space dual T^{-1*} of T^{-1} is an isometry of $L^\infty(\phi)$. Thus $T^{-1*}(x)=w\alpha(x)$, $x \in M$, where w is unitary in M and α is a C^*-automorphism of M. There is an isometry S of $L^1(\phi)$ such that $S^*=\alpha$. Thus if $x\in M$, $y \in M$, then $\langle T^{-1}(x), y\rangle = \langle x, T^{-1*}(y)\rangle = \langle x, w\alpha(y)\rangle = \langle xw, \alpha(y)\rangle = \langle S(xw), y\rangle$. Hence

$$(1) \qquad T^{-1}(x) = S(xw), \qquad x \in M.$$

Using [1, Théorème 2] there is a positive operator z affiliated with the center of M such that $x \rightarrow z\alpha(x)$ acts as an L^2-isometry on M. Thus if $x \in M$, then z and $\alpha(x)z$ belong to $L^2(\phi)$, so by Hölder's inequality [8, Corollary 12.9], $\alpha(x)z^2$ belongs to $L^1(\phi)$. We assert that for $x, y \in M$, $\langle \alpha(x)z^2, \alpha(y)\rangle = (z\alpha(y) \mid z\alpha(x^*))$. Indeed, this is trivial if z belongs to M. Otherwise, write $z = \int_0^\infty \lambda de_\lambda$ where $e_\lambda \in Z$ [3, p. 17]. Then $z_n \equiv \int_0^n \lambda de_\lambda \in Z$ and it is easy to check that $z_n z = z z_n$ and $\|z_n - z\|_2 \rightarrow 0$ (cf. [8, Corollary 12.13]). Hence

$$\|z^2 - z_m^2\|_1 = \|(z+z_m)(z-z_m)\|_1 \leqq \|z+z_m\|_2 \|z-z_m\|_2 \rightarrow 0.$$

Thus

$$\langle \alpha(x)z^2, \alpha(y)\rangle = \lim_n \langle \alpha(x)z_n^2, \alpha(y)\rangle$$

$$= \lim_n (z_n\alpha(y) \mid z_n\alpha(x^*)) = (z\alpha(y) \mid z\alpha(x^*))$$

proving the assertion. Now if $x, y \in M$,

$$\langle S(\alpha(x)z^2), y\rangle = \langle \alpha(x)z^2, \alpha(y)\rangle = (z\alpha(y) \mid z\alpha(x^*)) = (y \mid x^*) = \langle x, y\rangle,$$

so that $S(\alpha(x)z^2)=x$, $x\in M$. Combining this with (1) yields $T(x) =\alpha(x)z^2w^{-1}$, $x\in M$, which proves the theorem.

If M is a von Neumann algebra we denote by M_h the real Banach space of selfadjoint operators in M, by M^+ the cone of positive operators in M, by M_P the lattice of projections in M, and by S_h the convex set of all selfadjoint operators in M of operator norm at most one.

THEOREM 2. *Let M be a von Neumann algebra with a faithful normal finite trace ϕ, and let T be a linear L^p-isometry of M onto M for some p, $1\leq p<\infty$. Then* (i) *T is a C^*-automorphism of M if, and only if, one of the following conditions is satisfied*:

(ii) *$T(M^+)\subset M^+$ and $T(I)=I$;*

(iii) *$T(M_P)\subset M_P$;*

(iv) *$T(S_h)\subset S_h$ and $T(I)=I$.*

COROLLARY 1. *In Theorem 2, if M is a factor then T is either a *-automorphism of M or a *-anti-automorphism of M.*

COROLLARY 2. *In Theorem 2, if M is a factor and $p=1$ or $p=2$ the assumption $T(I)=I$ may be dropped in condition* (ii).

PROOF OF THEOREM 2. (i)$\Rightarrow$(iv). This is known [5, Theorem 5].

(iv)$\Rightarrow$(iii). We may assume that $\phi(I)=1$. If u is selfadjoint and unitary in M, then $t = T(u)$ is selfadjoint, $\|t\|\leq 1$ and $\phi(|t|^p)^{1/p}=\|t\|_p =\|u\|_p=1$. Thus $\phi(I-|t|^p)=0$ so that t is unitary. Now if $e\in M_P$, then $I-2e$ is selfadjoint and unitary, $I-2T(e)$ is selfadjoint and unitary, so that $T(e)\in M_P$.

(iii)$\Rightarrow$(ii). Note first that T is bounded in the L^∞-norm. This follows from the closed graph theorem and the identity $\|x\|_p\leq\|x\|$, $x\in M$. Next $T(I)\in M_P$, say $T(I)=e$, and $1=\|I\|_p=\|e\|_p=\phi(e^p)^{1/p} =\phi(e)^{1/p}$. Hence $\phi(I-e)=0$ which implies that $e=I$. Now let $a\in M^+$. By the spectral theorem a is the limit in L^∞-norm of operators b_j of the form $b_j=\sum_{i=1}^{n_j}\lambda_i e_i$ where $\lambda_i\geq 0$ and $e_1, \cdots, e_{n_j}$ are orthogonal projections in M. Since $T(b_j)$ belongs to M^+, so does $T(a)$.

(ii)$\Rightarrow$(i). By [7, Corollary 1], T has L^∞-norm 1. Thus if u is unitary in M and $t=T(u)$, then $\|t\|\leqq 1$, $\|t\|_p=1$, so that $\phi(I-|t|^p)=0$ which implies that t is unitary. The result now follows from [7, Corollary 2].

The proof of Corollary 2 rests on the following

LEMMA. *Let M be a von Neumann algebra with a faithful normal semifinite trace ϕ, and let T be an L^p-isometry of M onto M for $p=1$ or $p=2$. If a, $b\in M^+\cap m_\phi$, and $ab=0$, then $T(a)T(b)=0$.*

PROOF. The case $p=2$ can be found in [1, Lemma 2]. Since $ab=0$ we have $\|a\pm b\|_1=\phi(|a\pm b|)=\phi((a^2+b^2)^{1/2})$. Thus $\|a-b\|_1=\|a+b\|_1$ $=\phi(a+b)=\phi(a)+\phi(b)=\|a\|_1+\|b\|_1$. The map $x\rightarrow f_x$, $x\in m_\phi$, where f_x is the linear functional $y\rightarrow\phi(xy)$ on M, is linear, selfadjoint, positive and norm preserving in the sense that $\|x\|_1=\|f_x\|$ [3, p. 105]. Thus

$$\begin{aligned}\|f_{T(a)}-f_{T(b)}\| &= \|f_{T(a-b)}\| = \|T(a-b)\|_1 = \|a-b\|_1 = \|a\|_1+\|b\|_1 \\ &= \|T(a)\|_1+\|T(b)\|_1 = \|f_{T(a)}\|+\|f_{T(b)}\|.\end{aligned}$$

By [4, p. 243], $f_{T(a)}$ and $f_{T(b)}$ have disjoint supports [3, p. 61]. It follows that $T(a)T(b)=0$.

PROOF OF COROLLARY 2. Since M is a factor it suffices to show that $T(I)$ commutes with $T(x)$ for all $x\in M$. We may assume x is a projection p. By the Lemma, $T(p)$ and $T(I)-T(p)$ have zero product which implies that $T(I)$ commutes with $T(p)$.

It is interesting to note that in the case $p=2$ of Theorem 2, condition (ii) cannot be weakened. The trivial example $T(x)=-x$ shows that we must assume $T(I)=I$. Furthermore, we can show the theorem to be false if (ii) is replaced by the weaker condition (ii′) $T(M_h)\subset M_h$ and $T(I)=I$. To see this suppose that an L^2-isometry of a finite factor M satisfying (ii′) is always a *-automorphism or a *-anti-automorphism. Let N be a subfactor of M. Using [2, Théorème 8] each element x in M has a unique decomposition $x=x_1+x_2$ where $x_1\in N$ and x_2 is an element of M of trace 0. If α is a *-automorphism of N, the mapping $\tilde{\alpha}(x)=\alpha(x_1)+x_2$ is a linear L^2-isometry of M satisfying (ii′), so according to our supposition is a *-automorphism of M. If for example we let N be the hyperfinite factor and we let M be the crossed product of N by a group G of order 2 of outer *-automorphisms of N (see [9]), then the above discussion implies that an arbitrary *-automorphism of N commutes with each *-automorphism of N of order 2, which is absurd.

4. Remarks. 1. The extension of Theorems 1 and 2 to the semifinite case is open. For $p=2$ this has been done by M. Broise [1] for conditions (i) and (ii) of Theorem 2.

2. The extension of Theorem 1 to the case $1<p<\infty$, $p\neq 2$, is open. If M is commutative and semifinite, this extension is known [6, Theorem 3.1].

3. The results of this paper should prove to be useful for attacking the extension problems of *-isomorphisms between subalgebras of von Neumann algebras and therefore for constructing outer *-automorphisms on factors of type II_1. We propose to investigate this in a subsequent paper.

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