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California State College at Long Beach and
Northwestern University

SOME PROPERTIES OF DISTRIBUTIONS WHOSE PARTIAL DERIVATIVES ARE REPRESENTABLE BY INTEGRATION[1]

BY HERBERT FEDERER

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We denote n dimensional Euclidean space by R^n and let H^m be m dimensional Hausdorff measure.

It is well known that distributions of the type described in the title may alternately be characterized as corresponding to H^n measurable real valued functions f with the following property: *There exists a sequence of infinitely differentiable real valued functions* f_j *on* R^n *such that*

$$\lim_{j\to\infty}\int_K |f_j - f|\,dH^n = 0 \quad \text{and} \quad \liminf_{j\to\infty}\int_K \|Df_j\|\,dH^n < \infty$$

for every compact subset K *of* R^n. The class of such functions f is now widely regarded as the proper generalization to $n>1$ of the class of those functions on R which are H^1 equivalent to functions with finite total variation on every compact interval. However up to now there has been lacking an extension to $n>1$ of the basic classical results describing the continuity properties of functions with locally finite variation, namely that the set of points of discontinuity is countable and that one-sided limits exist everywhere. At first sight such an

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extension to $n>1$ seems impossible, because f may be H^n essentially unbounded, and every function H^n equivalent to f may be discontinuous everywhere. The following new results show that the problem can be solved through use of H^n approximate upper and lower limits, and the $n-1$ dimensional measure H^{n-1}.

For each $x \in R^n$ we let $\lambda(x)$ and $\mu(x)$ be the lower and upper H^n approximate limits of f at x; we recall that $\lambda(x)$ equals the least upper bound (possibly ∞ or $-\infty$) of the set of all real numbers t such that

$$\lim_{r\to 0+} r^{-n}H^n[R^n \cap \{z: |z-x| \leqq r, f(z) \leqq t\}] = 0.$$

Defining

$$\begin{aligned} A &= \{x: |\lambda(x)| = \infty \quad \text{or} \quad |\mu(x)| = \infty\}, \\ B &= \{x: -\infty < \lambda(x) < \mu(x) < \infty\}, \\ g(x) &= [\lambda(x) + \mu(x)]/2 \quad \text{for } x \in R^n \sim A, \end{aligned}$$

and asuuming $n>1$ we prove:

(1) $H^{n-1}(A)=0$.

(2) *H^{n-1} almost all of B can be covered by a countable family of $n-1$ dimensional regular, proper submanifolds of class 1 of R^n.*

(3) *For H^{n-1} almost all x in $R^n \sim B$,*

$$\lim_{r\to 0+} r^{-n}\int_{\{z:\ |z-x|\leqq r\}} |f(z) - g(x)|^{n/(n-1)}dH^n z = 0.$$

(4) *For H^{n-1} almost all x in B there exists a real valued linear function α on R^n such that*

$$\lim_{r\to 0+} r^{-n}\int_{P(r)} |f(z) - \lambda(x)|^{n/(n-1)}dH^n z = 0$$

with $P(r) = \{z: |z-x| \leqq r$ and $\alpha(z-x) \geqq 0\}$, and

$$\lim_{r\to 0+} r^{-n}\int_{N(r)} |f(z) - \mu(x)|^{n/(n-1)}dH^n z = 0$$

with $N(r) = \{z: |z-x| \leqq r$ and $\alpha(z-x) \leqq 0\}$.

(5) *For H^{n-1} almost all x in R^n,*

$$g(x) = \lim_{r\to 0+} \int_{R^n} f(x) r^{-n} k(r^{-1}|z-x|)dH^n z$$

whenever k is a real valued H^1 measurable function on R with compact support such that

$$\int_{R^n} k(|z|)dH^n z = 1 \quad \textit{and} \quad \int_{R^n} |k(|z|)|^n dH^n z < \infty.$$

The proposition (5) exhibits g as a very attractive representative of the H^n equivalence class of f, because arbitrary rotationally symmetric regularizations of f converge to g not only H^n almost everywhere, but H^{n-1} almost everywhere! (The corresponding proposition for $n=1$ asserts convergence H^0 almost everywhere, which means everywhere, and is classical; in this case g has locally finite total variation and $\lambda(x)$, $\mu(x)$ equal the one-sided limits of g at x.) There remains the problem of generalizing (5) to certain functions k with noncompact support, in particular to determine the smallest exponent needed to reproduce g from its Fourier transform H^{n-1} almost everywhere by spherical Riesz-Bochner summation (see [1]), provided f behaves suitably at infinity. The rotationally symmetric approach is necessary because the directions of the half-spaces occurring in (4) depend on x.

Proofs of the above propositions will appear in the author's forthcoming book "*Geometric measure theory*". Our arguments make use of the set

$$H = (R^n \times R) \cap \{(x, y) : y \leqq \mu(x)\},$$

whose essential boundary (for the purpose of the Gauss-Green theorem) is H^n almost equal to the set

$$C = (R^n \times R) \cap \{(x, y) : \lambda(x) \leqq y \leqq \mu(x)\},$$

which can H^n almost be covered by a countable family of n dimensional regular, proper submanifolds of class 1 of $R^n \times R$. From the geometric properties of these sets we obtain analytic properties of f by studying the n dimensional normal current T in R^n (see [3]) defined by the formula

$$T(\phi) = \int_{R^n} f \wedge \phi$$

for every infinitely differentiable differential form ϕ of degree n on R^n with compact support. For example, we show that

$$\|\partial T\|(W) = \int_R H^{n-1}[W \cap \{x : \lambda(x) \leqq y \leqq \mu(x)\}]dH^1 y$$

for every Borel subset W *of* R^n. (For the special case when f is linearly continuous in the sense of [4], this formula has been obtained independently by a different method in [5].) Our approach also yields a new proof of the following result of [2]: *For H^n almost all x in R^n there exists a real valued linear function β on R^n such that*

$$\lim_{r\to 0+} r^{-n} \int_{\{z:\ |z-x|\leq r\}} \epsilon(z)^{n/(n-1)} dH^n z = 0$$

with $\epsilon(z) = |f(z) - f(x) - \beta(z-x)| / |z-x|$.

The author wishes to thank Casper Goffman for some stimulating discussions about the subject of this note, in particular for raising questions concerning the nature of the sets A, B and C. Regarding the class of functions studied in [4] one can now assert the *equivalence of the four properties*: (I) *f is essentially linearly continuous.* (II) *g is linearly continuous.* (III) $H^{n-1}(B) = 0$. (IV) $\|\partial T\|(W) = 0$ *whenever* $W \subset R^n$ *with* $H^{n-1}(W) < \infty$.

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Brown University