STRONGLY CONVEX METRICS IN CELLS[1]

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The following question was raised by Bing in [2]: "If an n-dimensional compact topological space has a metric which is strongly convex and without ramifications (defined below), is it necessarily homeomorphic to the Euclidean n-cell?" Lelek and Nitka [5] answered this affirmatively for $n \leqq 2$; we outline below a proof that the answer is also yes when $n=3$. Although the question remains open in higher dimensions, we also give an affirmative answer when the space is assumed to be a manifold (= manifold with boundary) and $n \neq 4$ or 5. In fact with this further assumption we may omit the "without ramifications" requirement when $n \leqq 3$.

If X is a space and $x, y, m \in X$, then m is called a *midpoint* of x and y (with respect to a metric d on X) if $d(x, m) = d(m, y) = \frac{1}{2}d(x, y)$. The metric is *strongly convex* (SC) if each pair of points has a unique midpoint and *without ramifications* (WR) if no midpoint of x and y is a midpoint of x' and y unless $x' = x$. Both of these properties are enjoyed by the usual metric on Euclidean spaces and cells, and they are preserved under cartesian products in the following sense:

PROPOSITION 1. *If d_i is a* SC (*or* WR) *metric on X_i, $i=1, \cdots, n$ then $d(x, y) = \sum [d_i(x_i, y_i)^2]^{1/2}$ determines a* SC (*resp.* WR) *metric on $X = X_1 \times \cdots \times X_n$. (Here x_i denotes the* ith *coordinate of $x = \{x_i\} \in X$ and the sum extends over $i = 1, \cdots, n$.)*

Indeed an easy exercise in inequalities verifies that $\{m_i\}$ is a midpoint of $\{x_i\}$ and $\{y_i\}$ in (X, d) iff each m_i is a midpoint of x_i and y_i in (X_i, d_i).

Strongly convex metrics. Joining any two points in a complete SC metric space, there is a unique arc (called a *segment*) which is isometric to a closed interval of the real line [7]. It follows that the intersection of any two segments is connected or empty. In a compact SC metric space, segments vary continuously with their endpoints, allowing one to imitate some of the tricks available in Euclidean space. For example, by moving points along segments toward a fixed basepoint we can obtain deformations of the space and prove (see [2])

[1] These results are a portion of the author's Ph.D. thesis, written under Joseph Martin at the University of Wisconsin.

PROPOSITION 2. *Each compact* SC *metric space is contractible and locally contractible (hence an absolute retract if finite-dimensional).*

Thus the only compact 1- or 2-manifolds having SC metrics are the cells.

There are, however, examples of absolute retracts which fail to admit SC metrics—e.g. finite 2-complexes which are contractible but not collapsible (see [8] and [10]). The following theorem implies that such examples exist among 3-manifolds if and only if the Poincaré conjecture is false.

THEOREM 1. *If a compact 3-manifold has a* SC *metric, it is a 3-cell.*

OUTLINE OF PROOF. Let d be a SC metric on the 3-manifold M and choose a fixed $p\in M$. By Proposition 2, M has the homotopy type of a 3-cell, and it follows that there exists a homeomorphism h of the unit sphere $S^2\subset R^3$ onto $\mathrm{Bd}(M)$. Adjoin $A=[0, \infty)\times \mathrm{Bd}(M)$ to M by identifying each $b\in \mathrm{Bd}(M)$ with $(0, b)\in A$. Now a continuous function $f\colon R^3\to M\cup A$ may be defined such that (i) $f(0)=p$, (ii) f maps an initial part of the ray from 0 through $s\in S^2$ isometrically onto the segment in M from p to $h(s)$ and (iii) f maps the remainder of this ray isometrically (w.r.t. the usual metric on $[0, \infty)$) onto $h(s)\times[0, \infty)$. Now if e is an endpoint of a maximal segment it is possible to shrink $M-e$ in itself to a point, and we conclude that $e\in \mathrm{Bd}(M)$. Thus f maps R^3 onto $M\cup A$. If $y\in M\cup A$, then either $f^{-1}(y)$ is a single point, or else $y\in M$ and a linking argument shows that $f^{-1}(y)$ is a nonseparating subcontinuum of the sphere in R^3 centered at 0 and having radius $d(p, y)$. Thus $M\cup A$ may be considered as a decomposition of R^3 by pointlike sets lying on concentric spheres. Methods of Dyer and Hamstrom (see [3, p. 116]) imply that there exists a homeomorphism g of $M\cup A$ onto R^3 such that, in fact, $\|g(m)\|=d(p, m)$ if $m\in M$. It follows that $g(M)$, and hence M, is a 3-cell.

A *crumpled cube* is a space homeomorphic to the closure of the bounded complementary domain of a 2-sphere in R^3. An almost identical proof yields

THEOREM 2. *Any crumpled cube having a* SC *metric is a 3-cell.*

Some interesting corollaries also follow from the proof.

COROLLARY 1. *Let d be an arbitrary* SC *metric on the unit 3-cell $B^3\subset R^3$. Then (1) each segment σ(w.r.t. d) in B is a tame arc in R^3 and (2) if $b\in B$ and $\epsilon>0$, the set $Q=\{x\in B\colon d(x, p)\leqq\epsilon\}$ is a tame 3-cell in R^3.*

Proof. Assume $b=p$, $B=M$ and $\mathrm{Cl}(R^3-B)=A$ in the above proof (Cl = closure), and suppose $p\in\sigma$. Then $g(Q)$ is a starlike 3-cell in R^3 and $g(\sigma)$ is an arc meeting each sphere centered at 0 in at most two points. Thus $g(\sigma)$ and $g(Q)$ are tame and so are σ and Q.

Corollary 2. *The statement,* "If X and Y are spaces having SC metrics and $X\subset Y$, then some SC metric on X extends to a SC metric on Y," *is false in general.*

We need only take Y to be a 3-cell and X a wild arc in its interior· This is in sharp contrast to a theorem of Bing [1] that if X and Y admit convex metrics and $X\subset Y$, then *any* convex metric on X extends to a convex metric on Y. (*Convex* means that any two points have *at least* one midpoint.)

The 3-cell characterization. Now we attack the problem that we mentioned first, i.e. to get a metric characterization of B^3 without assuming that it be a manifold. It is not difficult to show that a complete convex metric space is simultaneously SC and WR iff whenever two segments meet in more than a point their union is a segment. Thus in the compact case each segment has a unique extension to a maximal segment and the deformations along segments described earlier are actually pseudo-isotopies. These facts and some elementary Vietoris homology are used in a proof (to appear in a later paper) of the following fundamental lemma.

Lemma 1. *Each finite-dimensional compact space X with a* SC—WR *metric has a dense open subset U such that $X-x$ fails to be contractible (in itself) whenever $x\in U$.*

Theorem 3. *If X is a 3-dimensional compact space with a* SC—WR *metric d, then X is homeomorphic to B^3.*

Outline of Proof. By the lemma there exist $p\in X$ and $\epsilon>0$ such that the set $N=\{x\in X: d(x, p)\leqq\epsilon\}$ contains no points with contractible complements. Hence no maximal segment ends in N. Let S be the boundary of N in X and let E be the set of endpoints of those maximal segments which hit p.

Using the fact that segments are so well behaved, one shows "geometrically" that (1) $N\cong \mathrm{C}(S)$ (C = cone), (2) S is a retract of $X-p$ and is therefore a compact absolute neighborhood retract, (3) S admits a fixed-point free "antipodal" homeomorphism, (4) no finite set separates S, (5) $\dim(N)=3$ and $\dim(S)=2$, (6) $S-s$ is contractible (in itself) whenever $s\in S$ and (7) S is a 1-1 continuous image of E. By (4), (5) and (6) we may conclude that $H_n(S)=0$ for $n\neq 2$

(H_* = reduced singular homology with integral coefficients). Then by (2) and (3) and a fixed-point theorem of Lefschetz [4, p. 116] $H_2(S) \neq 0$. But (6) implies that $H_n(S-s)=0$ for all $s \in S$ and $n \geqq 0$. These last two facts imply, by McCord's characterization [6], that S is a 2-sphere. By shrinking X along segments toward p, we can obtain an embedding of X in Int (N), which by (1) is homeomorphic to R^3. An invariance of domain argument shows that E is the boundary of X in any embedding in R^3, so E is compact. Then $X \cong C(E)$ and by (7) E is a 2-sphere, proving that X is a 3-cell.

Manifolds of higher dimension.

LEMMA 2. *Suppose M is a compact n-manifold with a* SC—WR *metric. Then* (1) $\text{Bd}(M) \neq \emptyset$ *and M is homeomorphic to the cone on* $\text{Bd}(M)$ *and* (2) *if $b \in \text{Bd}(M)$ then $\text{Bd}(M)-b$ is contractible in itself.*

PROOF. Choose $p \in \text{Int}(M)$ and let E be the set of endpoints of maximal segments through p. To verify (1) we need only show that $E=\text{Bd}(M)$. But if $e \in E$, then M and $M-e$ are both contractible spaces, so the relative homology group, $H_n(M, M-e)$ is trivial. Hence $e \in \text{Bd}(M)$. Now if $x \in \text{Bd}(M)-E$, there is a point q and a segment from p to q with x in its interior. By pulling M along segments toward q, one obtains a homeomorphism of M into M taking p to $x \in \text{Bd}(M)$, which is impossible. To show (2), let $r: M-p \to \text{Bd}(M)$ be the retraction outward along segments. Let $b' \in \text{Bd}(M)$ be the other endpoint of the maximal segment through b and p. Pulling toward b' along segments defines a homotopy h_t of $\text{Bd}(M)$ in M such that h_0 = identity and $h_1(\text{Bd}(M))=b'$. By the WR property, $h_t(\text{Bd}(M)-b)$ misses the segment from p to b, so $rh_t | \text{Bd}(M)-b$ is a contraction of $\text{Bd}(M)-b$ in itself.

The second part of the lemma implies that $\text{Bd}(M)$ is an $(n-2)$-connected closed $(n-1)$-manifold. Theorem 4 is then an application of the recent solution of the topological Poincaré conjecture in dimensions other than 3 and 4. F. Taranzos [9] has recently announced a proof of Theorem 4 without restriction on n, presumably by different methods.

THEOREM 4. *Each compact n-manifold ($n \neq 4$ or 5) having a* SC—WR *metric is an n-cell.*

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