NORM CHARACTERIZATION OF REAL L^p SPACES[1]

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It is well known that a real Banach space, X, is a Hilbert space if and only if its norm satisfies the parallelogram law:

$$\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2.$$

In this note we announce a norm characterization of real L^p space in terms of Clarkson's inequalities [1]; and the norm-1 productions on X

(1) $$\|x+y\|^p+\|x-y\|^p\leqq 2\|x\|^p+2\|y\|^p \quad \text{if } 1<p\leqq 2.$$

(2) $$\|x+y\|^q+\|x-y\|^q\geqq 2\|x\|^q+2\|y\|^q \quad \text{if } 2\leqq q<\infty.$$

We outline below the two main steps in our proof. Detailed proof, applications, and generalizations concerning Orlicz spaces determined by strictly convex Young's functions will be published elsewhere.

1. **Representation lemma.** The representation lemma stated below is based on techniques developed by Cunningham [2], for the case $p=1$. Since the cases for $p=1$ and $p=2$ are known, we assume throughout the following that $1<p<2$. A projection, E, on X is an L^p projection iff

$$\|x\|^p=\|Ex\|^p+\|(I-E)x\|^p$$

for all $x\in X$. $P(X)$ denotes the class of all L^p projections on X. $P(X)$ is clearly nonempty because $\{0, I\}\subseteq P(X)$; and if (1) holds in X then $P(X)$ is a complete Boolean algebra of norm 1 projections under the usual order. An element $u\in X$ is said to be an L^p unit iff

$$\mathrm{Cl}\ (\mathrm{span}\ \{Eu \mid E\in P(X)\})=X.$$

If X has an L^p unit then X is linearly isometric to $L^p(S, \Sigma, \mu)$, where S is the Stone-space of $P(X)$, Σ the σ-ring generated by the closed-open subsets of S, and μ is given by $\mu(E)=\|Eu\|^p$. Following the notation of Cunningham, let

$$S_x\equiv \mathrm{Cl}(\mathrm{span}\{Ex \mid E\in P(X)\}).$$

Then x is said to be a local L^p unit if S_x is the range of some $E\in P(X)$.

[1] These results are part of the author's doctoral dissertation at the University of Pittsburgh. The author wishes to thank his advisor, Professor Henry B. Cohen, for suggesting the use of Clarkson's inequality.

LEMMA. *X is linearly isometric to a real L^p space for some fixed p, $1<p<2$, iff (1) holds in X and every $x\in X$ is a local L^p unit.*

2. **Characterization.** If (1) holds in X, and X^* satisfies the dual inequality (2) for $q=p/(p-1)$, then $E\in P(X)$ iff $E^*\in Q(X^*)$ (i.e., $P(X)^*\subseteq Q(X^*)$). Clarkson's results show that both X and X^* are uniformly convex. Hence, by the theorem of James [3], the Gateaux differential

$$N(x;y) \equiv \lim_{h\to 0} (\|x+hy\| - \|x\|)/h$$

exists for all $x, y\in X$ and $N(x;\cdot)$ is a linear functional of norm 1 for each $x\in X$. Since uniform convexity implies reflexity the assumption that $p<2$ in the above lemma is removed. Furthermore the duality mapping,

$$\phi: x \to N(x;\cdot)\|x\|^{p-1}; \qquad \phi(0)\equiv 0$$

is 1-1 from X onto X^*. The orthogonality properties of the Gateaux differential imply that for each $x\in X$, S_x is the range of a norm 1 projection, T, on X.

THEOREM. *Let X be an arbitrary real Banach space and p an arbitrary but fixed real number $1<p<\infty$; then X is linearly isometric to a real $L^p(S,\Sigma,\mu)$, where S is a set, Σ is a σ-ring of subsets of S, and μ is a measure on S, iff X satisfies Clarkson's inequality for p, and X^* satisfies the dual inequality for $q=p/(p-1)$; and $N(Ex;y)=0$ for all $E\in P(X)$ implies $\|x+y\|^p=\|x\|^p+\|y\|^p$.*

REFERENCES

1. J. A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc. **40** (1936), 396–414.

2. F. J. Cunningham, Jr., *L-structure in L-spaces*, Trans. Amer. Math. Soc. **95** (1960), 274–299.

3. R. C. James, *Orthogonality and linear functionals in normed linear spaces*, Trans. Amer. Math. Soc. **61** (1947), 265–292.

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