CHARACTERIZATIONS OF FAVARD CLASSES FOR FUNCTIONS OF SEVERAL VARIABLES

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1. Introduction. There are two general approaches to the study of saturation theory (for the definitions see [5]), namely the integral transform method (cf. [3], [5]) and the semigroup method (cf. [4]). In this note a third method, a distribution theoretical method will be employed, in particular to characterize the Favard (saturation) classes defined by

$$(1)\qquad V_\alpha^p = \{f; f(x) \in L^p(E^n),\ |v|^\alpha f^\wedge(v) = g^\wedge(v),\ g \in L^p(E^n)\}.$$

Here x, v denote vectors in E^n with $|v| = (v_1^2 + \cdots + v_n^2)^{1/2}$ and α a positive parameter. $f^\wedge(v)$ being the Fourier transform of f, this definition of V_α^p is meaningful only for $1 < p \leqq 2$. In order to extend it to $2 < p < \infty$ we use the classes of Bessel potentials

$$(2)\ L_\alpha^p = \{f; f \in L^p(E^n),\ (1 + |v|^2)^{\alpha/2} f^\wedge(v) = h^\wedge(v),\ h \in L^p(E^n)\}.$$

This definition bears sense not only for $1 < p \leqq 2$ but also for $2 < p < \infty$ if the Fourier transform is taken in the distribution theoretical sense since $(1 + |v|^2)^{\alpha/2}$ is an infinitely differentiable and slowly increasing function (in the terminology of L. Schwartz [10]). The problem is to show the equivalence of (1) and (2) for $1 < p \leqq 2$ and to give simple characterizations of (2), e.g. in terms of differentiability properties both in the classical and the distributional (or Sobolev) sense. An equivalent definition of the classes L_α^p, investigated in [1], [2], [6], is given by

$$(3)\qquad L_\alpha^p = \{f; f \in L^p(E^n), f = G_\alpha * h,\ h \in L^p(E^n)\},$$

where $G_\alpha(x)$ is defined through $G_\alpha^\wedge(v) = (1 + |v|^2)^{-\alpha/2}$, having the properties $G_\alpha(x) \in L^1(E^n)$, $G_\alpha(x) \geqq 0$, $\int_{E^n} G_\alpha(x)dx = (2\pi)^{n/2}$. L_α^p is a subspace of the space of tempered distributions.

For integral values of α we obtain an equivalence between L_α^p and the Sobolev space

$$(4)\quad W_\alpha^p = \Big\{f; f \in L^p(E^n),\ D^k f \in L^p(E^n) \quad \text{for every } k = (k_1, \cdots, k_n) \text{ with } k_j \geqq 0 \text{ and } |k| = \sum_{j=1}^n k_j \leqq \alpha\Big\}.$$

Here $D^k f = \partial^{|k|} f / \partial x_1^{k_1} \cdots \partial x_n^{k_n}$ means the distribution derivative of f.

2. **Characterizations for $1 < p < \infty$, $\alpha > 0$.**

THEOREM 1. *For $1 < p \leqq 2$ and $\alpha > 0$, $f \in V_\alpha^p$ if and only if $f \in L_\alpha^p$.*

The proof depends upon a lemma in a paper of E. M. Stein [11] where also some further characterizations of the class L_α^p are given for $0 < \alpha < 2$. Theorem 1 states that L_α^p is a continuation of V_α^p for $p > 2$. Thus it is sufficient to give equivalent characterizations of L_α^p. As a second step, using known results of A. P. Calderón [6], N. Aronszajn, F. Mulla and P. Szeptycki [2] we have

THEOREM 2. *Let $\alpha = 1, 2, \cdots$ and $1 < p < \infty$. Then $f \in L_\alpha^p$ if and only if $f \in W_\alpha^p$.*

Theorems 1, 2 enable us to prove many other characterizations of V_α^p and L_α^p for special values of α or p, especially those given in terms of ordinary derivatives. The cases $\alpha = 1,2$ are the most important examples in the applications to saturation theory.

3. **The case $\alpha = 2$.** The following list of equivalences is a consequence of Theorems 1,2 and of results of R. J. Nessel [8].

THEOREM 3. *Let $f \in L^p(E^n)$, $1 < p < \infty$. The following assertions are equivalent:*

(a) *$f \in V_2^p$ (here the definition of V_2^p is extended for $p > 2$ by $|v|^2 f^\wedge(v) = g^\wedge(v)$, $g \in L^p(E^n)$, in the distributional sense);*

(b) *$f \in L_2^p$;*

(c) *$f \in W_2^p$;*

(d) *$\Delta f = g$, $g \in L^p(E^n)$ ($\Delta f = \partial^2 f / \partial x_1^2 + \cdots + \partial^2 f / \partial x_n^2$ in the distributional sense);*

(e) *for $j, k = 1, 2, \cdots, n$ the functions f, $\partial f / \partial x_j$ are absolutely continuous in each variable, and $\partial f / \partial x_j$, $\partial^2 f / \partial x_j \partial x_k \in L^p(E^n)$ (the derivatives to be understood in the ordinary sense);*

(f) *for $j = 1, 2, \cdots, n$ (with e_j = unit vector in j-direction)*

$$\|f(x + 2he_j) - 2f(x + he_j) + f(x)\|_p = O(h^2) \qquad (h \in E^1,\ h \to 0);$$

(g) $\|f(x+2u) - 2f(x+u) + f(x)\|_p = O(|u|^2) \qquad (u \in E^n,\ |u| \to 0)$;

(h) $\|\sum f(x+jh) - 2^n f(x)\|_p = O(h^2) \qquad (h \to 0)$,

where the sum runs over all $j = (j_1, \cdots, j_n)$ with $j_k = \pm 1$.

4. **The case $\alpha = 1$.** The Hilbert transform of a function $f \in L^p(E^n)$, $1 < p < \infty$, with respect to x_j is defined by

$$(5)\qquad \tilde{f_j}(x) = \lim_{\epsilon\to 0+} \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \int_{|x-u|\geq\epsilon} f(u) \frac{x_j - u_j}{|x-u|^{n+1}}\, du \qquad (j = 1, 2, \cdots, n).$$

The functions $f_j^\sim(x)$ form the coordinates of a vector $(Hf)(x) = \sum_{j=1}^n e_j f_j^\sim(x)$ (cf. [7]) and again belong to $L^p(E^n)$. We have, in the case $\alpha=1$,

THEOREM 4. *Let $f\in L^p(E^n)$, $1<p<\infty$. The following statements are equivalent*:

(a) $f\in L_1^p$,

(b) $f_j^\sim\in L_1^p$ for $1\leq j\leq n$,

(c) $f\in W_1^p$,

(d) $(\operatorname{div} Hf)(x) = \sum_{j=1}^n(\partial/\partial x_j)f_j^\sim(x)\in L^p(E^n)$ *(distributional derivatives)*,

(e) *for $j=1, 2, \cdots, n$ the functions $f_j^\sim(x)$ are absolutely continuous in each variable, and the ordinary derivatives $\partial f_j^\sim/\partial x_k$ $(k=1, \cdots, n)$ are in $L^p(E^n)$*,

(f) *for $j, k=1, 2, \cdots, n$,*

$$\|\tilde{f_j}(x + he_k) - \tilde{f_j}(x)\| = O(|h|) \qquad (h\to 0);$$

(g) $\|f(x+he_j)-f(x)\| = O(|h|)$ $h\to 0; j=1, 2, \cdots, n)$.

5. **The case** $p=2$. For $p=2$ many known equivalences between L_α^2 and "fractional" Sobolev spaces or various types of Besov spaces can be applied to give new characterizations of V_α^2 for arbitrary $\alpha>0$. For instance the fractional Sobolev spaces defined for fractional $\alpha=m+\beta$ $(m=0, 1, 2, \cdots; 0<\beta<1)$ by

$$W_\alpha^2 = \left\{f; f\in W_m^2, \left[\int_{E^n} \frac{\|D^k f(\cdot + u) - D^k f(\cdot)\|_2^2}{|u|^{2\beta+n}}\, du\right]^{1/2} < \infty \right.$$
$$\left. \text{for every } k = (k_1, \cdots, k_n) \text{ with } |k| = m\right\}$$

and for $\alpha=1, 2, \cdots$ by (4) (see [9], [2, p. 74]) are equivalent to V_α^2 for $\alpha>0$.

The present results are also connected with the theory of intermediate spaces which is presented in [4]. A detailed discussion of these results, including proofs and various extensions, will be published elsewhere.

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