IMMERSIONS OF G-MANIFOLDS, G FINITE

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G denotes a finite group. If G acts on X, and $H \subset G$, $X_H = \{x;\ hx = x,\ h \in H\}$.

1. P.L. **G-manifolds.** A *G-polyhedron* is a polyhedron K together with a P.L. action of G on K: in particular a *P.L. G-manifold* is a G-polyhedron whose polyhedron is a manifold. *Maps, subspaces* of G-polyhedra are G invariant maps, subspaces of the underlying polyhedra. A *Euclidean G space* is the P.L. G-manifold underlying a finite dimensional complex representation of G. A *G ball (pair)* is an invariant ball (pair) in some Euclidean G space. A P.L. G-manifold is *locally-Euclidean (l.e.)* if it has a covering by open sets each isomorphic to an open set in a G ball. A pair (N, M), N a G-manifold and M an unbounded submanifold contained in int N, is *locally Euclidean* if at each point p of M it is like a stabilizer p ball pair.

The *regular neighbourhood theorem* [4], [9] holds for l.e. G-manifolds but not in general. For example let S be a Whitehead sphere [8] and B the star of a fixed vertex: CS the cone on S, collapses to CB, but the two are distinct G-manifolds.

If P is a G-polyhedron and K a triangulation of P in which G acts by vertex permutation, a *G block bundle* over P will mean a block bundle ξ over K (see [5]) and an action of G on ξ as a group of bundle automorphisms compatible with the inclusion of K in the total space $E(\xi)$ such that for each simplex δ of K and block β above δ, $(\beta, \delta) \approx (B \times \delta, \delta)$ as H spaces, for some H ball B, where $H =$ stabilizer δ. $E(\xi)$ is naturally a G polyhedron. If P is a l.e. unbounded G-manifold $(E(\xi), P)$ is a l.e. pair and conversely

THEOREM 1. *Let $(N^n, M^m_n$ be a l.e. unbounded G-manifold and unbounded submanifold and suppose M is compact. $\exists n-m$ G block bundle ξ over M unique up to isomorphism and an embedding f: $E(\xi) \rightarrow N$ extending the inclusion of M. If g: $E(\xi) \rightarrow N$ is another such $\exists$ isotopy F_t of N mod M and an automorphism α of ξ with $g = F_1 \cdot f \cdot E(\alpha)$.*

2. P.L. **G-embeddings.** M and N will denote P.L. G-manifolds, M compact and both without boundary.

$E_G(M, N)$, $I_G(M, N)$, Homeo$_G(N)$ are the *semisimplicial complexes of embeddings of M in N, immersions of M in N, homeomorphisms of N.* A k simplex of Homeo$_G(N)$ is a G-homeomorphism of $\Delta^k \times N$ commuting with the projection onto Δ^k, where Δ^k is the standard k simplex on which G acts trivially and G acts on $\Delta^k \times N$ as a product. A k simplex of $E_G(M, N)$ is a G-embedding f of $\Delta^k \times M$ in $\Delta^k \times N$ commuting with the projections to Δ^k and such that $\exists$ an open covering $\{U_\alpha \times V_\alpha;\ U_\alpha \subset \Delta,\ V_\alpha \subset X\}$ of $\Delta^k \times M$, embeddings $g_\alpha\colon V_\alpha \to Y$, open sets W_α in Y containing image g_α and embeddings $h_\alpha\colon U_\alpha \times W_\alpha \to U_\alpha \times Y$ commuting with the projection to U_α, satisfying $f/U_\alpha \times V_\alpha = h_\alpha \cdot (\mathrm{id} \times g_\alpha)$. A k simplex of $I_G(M, N)$ is defined by replacing "embedding" by "immersion" in the definition of $E_G(M, N)$. An embedding $i\colon M \to N$ induces

$$i'\colon \mathrm{Homeo}_G(N) \to E_G(M, N),$$

$$\{\Delta^k \times N \xrightarrow{g} \Delta^k \times N\} \to \{\Delta^k \times M \xrightarrow{g\cdot(\mathrm{id} \times i')} \Delta^k \times N\}.$$

THEOREM 2. *i' is a fibration.*

Theorem 2 extends to the case of M bounded when the boundary is locally collared, and is proved by the method of [2], [3].

3. P.L. G-**immersions.** M and N will be as in (2), and l.e. G acts on the tangent micro-bundles $T(M)$ and $T(N)$ via the product actions on $M \times M$ and $N \times N$.

$\mathrm{Rep}_G(T(M), T(N))$ is a semisimplicial complex. A k simplex is a G invariant bundle map $f\colon \Delta^k \times T(M) \to \Delta^k \times T(N)$ commuting with the projections to Δ^k and satisfying

(a) the restriction of f to $f_1\colon \Delta^k \times M \to \Delta^k \times N$ is of codimension >0, i.e. for some point $p \in \Delta^k$ and each subgroup $H \subset G$, f_1 maps each component of $p \times M_H$ into a component of $p \times N_H$ of strictly higher dimension.

(b) the restriction f_2 of f to a fibre above Δ^k, $f_2\colon T(M) \to T(N)$, is "locally integrable," i.e. $\exists$ open covering $\{U_\alpha\}$ of M, open sets V_α in N, maps $g_\alpha\colon U_\alpha \to V_\alpha$ and bundle maps $h_\alpha\colon T(V_\alpha) \to T(N)$, satisfying $f_2/T(U_\alpha) = h_\alpha \cdot dg_\alpha$, where dg_α denotes the differential of g_α.

(c) $\exists$ open covering $\{U_\alpha \times V_\alpha;\ U_\alpha \subset \Delta,\ V_\alpha \subset M\}$ of $\Delta \times M$, micro bundles ν_α above V_α, bundle maps $g_\alpha\colon T(V_\alpha) \to \nu_\alpha$ and bundle isomorphisms $h_\alpha\colon U_\alpha \times \nu_\alpha \to \mu / U_\alpha \times V_\alpha$, where μ is the bundle induced over $\Delta^k \times M$ by f from $T(N)$, satisfying $f^*/U_\alpha \times T(V_\alpha) = h_\alpha \cdot (\mathrm{id} \times g_\alpha)$, where $f^*\colon \Delta^k \times T(M) \to \mu$ is induced from f. Let $I_G^+(M, N)$ denote the subcomplex of $I_G(M, N)$ satisfying (a) also.

THEOREM 3. *α: $I_G^+(M, N) \to \mathrm{Rep}_G(T(M), T(N))$ is a homotopy equivalence, the map α being the differential.*

The proof of Theorem 3 uses an extension of Theorem 2 together with the case G trivial (Haefliger-Poenaru [1]).

4. Smooth G-immersions. M and N will be smooth G-manifolds (of finite dimension) (see [7]), N without boundary, and M compact, and X a compact G space with X/G finite-dimensional.

Let $I=\text{Imm}^\infty(M, N)$, and $R=\text{Rep}^\infty(T(M), T(N))$ denote the spaces of smooth immersions of M in N, smooth representations of $T(M)$ in $T(N)$, respectively. G acts on I (and similarly on R): $f^g(m) = g^{-1}f(gm)$, for $f\in I$, $g\in G$, $m\in M$. If α: $X\to I$ or R is G invariant call α of codimension >0 if the induced mappings α_x: $M\to N$, or $T(M)\to T(N)$ are stabilizer x-mappings of codimension >0 in the sense of (3) for each $x\in X$. The differential induces a mapping d_X from the space of G invariant mappings $X\to I$ of codimension >0 to those $X\to R$ of codimension >0.

THEOREM 4. *d_X is a bijection on homotopy classes.*

There is a relative form of the theorem for pairs (X, A) where A is a closed G-subspace and X is as before.

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