INTERMEDIATE EXTENSIONS IN L^p

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1. **Introduction.** Let $A(x, D)$ be an elliptic operator defined on Euclidean n-dimensional space and let $q(x)$ be a locally square integrable function. Let A_0 and B_0 denote the operators $A(x, D)$ and $A(x, D)+q(x)$ acting on the set C_0^∞ of infinitely differentiable functions, respectively. Under suitable regularity conditions on the coefficients of $A(x, D)$ the minimal and maximal closed extension of A_0 in L^p coincide for $1<p<\infty$. Without further restrictions on q, this is not true for B_0.

The purpose of the present investigation is to find sufficient conditions on q such that some closed extension of B_0 will have the same essential spectrum as the closure A of A_0. For $p=2$ we found it convenient in [11] to employ regularly accretive extensions introduced by Kato [13]. However, this theory employs Hilbert space structure and is unapplicable for $p\neq 2$. Moreover, some of the L^2 estimates employed in [11] have no known counterparts in L^p for $p\neq 2$.

Our approach has been to develop a theory of extensions in Banach space which generalizes Kato's development. We call such operators "intermediate extensions." Under suitable conditions on $q(x)$ we are able to show that these extensions have the desired properties.

2. **Intermediate extensions.** Let A_0 be a densely defined, preclosed linear operator from a Banach space X to a Banach space Y. Then $D(A_0^*)$ is weakly* dense in Y^*. Let S be a linear manifold in $D(A_0^*)$ which is also weakly* dense in Y^*. We consider all closed extensions A of A_0 such that $D(A^*)\supseteq S$. The closure $\bar{A}$ of A_0 is the smallest such extension and therefore will be called the *minimal* extension of A_0. There is a largest such extension $\tilde{A}$. $D(\tilde{A})$ consists of those $u\in X$ for which there is an $f\in Y$ satisfying

$$(u, A_0^* v) = (f, v) \qquad \text{for all } v \in S.$$

We then set $\tilde{A}u=f$. This operator is well defined, for if $(u, A_0^*v) = (g, v)$ for all $v\in S$, then $(f-g, v)=0$ for all such v. Since S is weakly* dense in Y^*, we have $f=g$. Moreover, if A is any closed extension of A_0 with $D(A^*)\supseteq S$, then

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$$(u, A^*v) = (Au, v) \qquad \text{for all } u \in D(A),\ v \in D(A^*).$$

In particular, this holds for $v \in S$, showing that $u \in D(\tilde{A})$ and $\tilde{A}u = Au$. We call $\tilde{A}$ the *maximal* extension of A_0 relative to S.

We now consider a method of obtaining other closed extensions of A_0 "between" the minimal and maximal extensions. We call such extensions *intermediate*. Let W be a Banach space containing $D(A_0)$ and continuously embedded in X. Similarly, let Z be a Banach space containing S and continuously embedded in Y^*.

DEFINITION 2.1. An operator A from X to Y will be called an intermediate extension of A_0 relative to W and Z (or briefly a W-Z extension of A_0) if

(a) There is a continuous bilinear form $a(u, v)$ on $W \times Z$ such that $a(u, v) = (A_0 u, v)$ on $D(A_0) \times Z$.

(b) $u \in D(A)$ and $Au = f$ if and only if $u \in W$, $f \in Y$ and $a(u, v) = (f, v)$ for all $v \in Z$.

Note that a W-Z extension need not be closed.

LEMMA 2.2. *If $D(A_0)$ is dense in W, then a necessary and sufficient condition that A_0 have a unique W-Z extension is that*

$$(2.1) \qquad |(A_0 u, v)| \leqq C\|u\|_W \|v\|_Z \qquad \text{for all } u \in D(A_0), \quad v \in Z.$$

REMARK 2.3. If $Z = Y^*$ and $W = D(\bar{A})$, then the W-Z extension A of A_0 is $\bar{A}$, the minimal extension. If $W = X$ and Z is the closure of S in $D(A_0^*)$, then $A = \tilde{A}$, the maximal extension.

THEOREM 2.4. *A necessary and sufficient condition that a W-Z extension A be closed is that*

$$(2.2) \qquad \|u\|_W \leqq C(\|Au\|_Y + \|u\|_X), \qquad u \in D(A).$$

THEOREM 2.5. *If*

$$(2.3) \qquad \|u\|_W \leqq C\Big(\sup_{v \in Z} |(A_0 u, v)| / \|v\|_Z + \|u\|_X\Big), \qquad u \in D(A_0),$$

then the W-Z extension A of A_0 is closed.

The class $\Phi(X, Y)$ of Fredholm operators from X to Y is defined as the set of densely defined, closed, linear operators from X to Y, having closed ranges with the dimensions of their null spaces and codimensions of their ranges both finite (cf. [1]). For $A \in \Phi(X, Y)$, the index $i(A)$ of A is the difference between the dimension of its null space and the codimension of its range.

THEOREM 2.6. *A W-Z extension A is in $\Phi(X, Y)$ if and only if*

$$\|u\|_W \leqq C\|Au\|_Y + \|u\|_K, \qquad u \in D(A) \tag{2.4}$$

and

$$\|v\|_Z \leqq C\|A^*v\|_{X^*} + \|v\|_{K'}, \qquad v \in D(A^*), \tag{2.5}$$

where the norm $\| \ \|_K$ (resp. $\| \ \|_{K'}$) *is completely continuous with respect to the norm of* W (resp. Z).

3. Definitions and main results. We shall be working in the $H^{s,p}$ spaces for s real and $1<p<\infty$. Let C_0^∞ denote the set of infinitely differentiable complex valued functions with compact supports in n-dimensional Euclidean space E^n. For $\phi \in C_0^\infty$ we define the norm

$$\|\phi\|_{s,p} = \left[\int |\mathfrak{F}^{-1}(1+|\xi|^2)^{s/2}\mathfrak{F}\phi|^p dx\right]^{1/p}, \tag{3.1}$$

where $\mathfrak{F}$ denotes the Fourier transform, $x=(x_1, \cdots, x_n)$ is the variable in E^n, $\xi=(\xi_1, \cdots, \xi_n)$ is the argument of $\mathfrak{F}\phi$ and $\mathfrak{F}^{-1}$ denotes the inverse Fourier transform. For s a nonnegative integer, the norm (3.1) is equivalent to the sum of the L^p norms of ϕ and all its derivatives up to order s. The completion of C_0^∞ with respect to the norm (3.1) is denoted by $H^{s,p}$.

Let $q(x)$ be a measurable function on E^n. For α real and $1<p<\infty$ we set

$$\begin{aligned} M_{\alpha,p}(q) &= \sup_x \int_{|x-y|<1} |q(y)|^p dy, \qquad \alpha > 0, \\ &= \sup_x \int_{|x-y|<1} |q(y)|^p (1-\log|x-y|)dy, \qquad \alpha = 0, \\ &= \sup_x \int_{|x-y|<1} |q(y)|^p |x-y|^\alpha dy, \qquad -n < \alpha < 0, \\ &= \sup_x |q(x)|, \qquad \alpha \leqq -n. \end{aligned}$$

We let $M_{\alpha,p}$ denote the set of those measurable q for which $M_{\alpha,p}(q) < \infty$.

We shall consider an elliptic operator

$$A(x, D) = \sum_{|\mu| \leq m} a_\mu(x) D^\mu \tag{3.2}$$

of order m on E^n, where $\mu=(\mu_1, \cdots, \mu_n)$ is a multi-index of nonnegative integers, $|\mu| = \mu_1 + \cdots + \mu_n$, and

$$D^\mu = (-i)^{|\mu|}\partial^{|\mu|}/\partial x_1^{\mu_1} \cdots \partial x_n^{\mu_n}.$$

We assume that $A(x, D)$ is uniformly elliptic in E^n. This means that there is a constant $C_0 > 0$ such that

$$\left| \sum_{|\mu|=m} a_\mu(x)\xi^\mu \right| \geqq C_0 |\xi|^m$$

for all points $x \in E^n$ and real vectors ξ, where

$$\xi^\mu = \xi_1^{\mu_1} \cdots \xi_n^{\mu_n}.$$

The formal adjoint of $A(x, D)$ is given by

$$A'(x, D) = \sum_{|\mu| \leq m} D^\mu(a_\mu(x)).$$

We assume that the coefficients of both $A(x, D)$ and $A'(x, D)$ are bounded and that for $|\mu| = m$ the $a_\mu(x)$ are continuous. All of the coefficients of both $A(x, D)$ and $A'(x, D)$ are to approach limits as $|x| \to \infty$.

Let A_0 denote $A(x, D)$ acting on C_0^∞ and considered as an operator on L^p. As is well known, A_0 is preclosed and all closed extensions having C_0^∞ in the domains of their adjoints coincide (cf., e.g., [2]). Set

$$A(\infty, D) = \sum_{|\mu| \leq m} a_\mu D^\mu,$$

where a_μ is the limit of $a_\mu(x)$ as $|x| \to \infty$.

Let A and A_∞ denote the closed extensions of A_0 and $A(\infty, D)$ acting on C_0^∞, respectively. The spectrum $\sigma(A_\infty)$ of A_∞ can be computed by means of Fourier transforms (cf., e.g., [3]). It is the set R_0 of values assumed by

$$A(\infty, \xi) = \sum_{|\mu| \leq m} a_\mu \xi^\mu$$

where ξ ranges over all real values. For $\lambda \in R_0$ the range of $A_\infty - \lambda$ is not closed in L^p.

For an arbitrary operator E in a Banach space X the Φ-set Φ_E of E is defined as the set of those complex λ for which $E - \lambda \in \Phi(X) \equiv \Phi(X, X)$. In the case of A_∞ we have

$$\Phi_{A_\infty} = \rho(A_\infty) = CR_0, \tag{3.3}$$

where CR_0 denotes the complement of R_0 in the complex plane. For the operator A we have

THEOREM 3.1.

$$\Phi_A = CR_0.$$

According to Wolf [4] the essential spectrum $\sigma_{ew}(E)$ of an operator E is defined as $C\Phi_E$. Thus another way of stating Theorem 3.1 is

$$\sigma_{ew}(A) = \sigma_{ew}(A_\infty) = \sigma(A_\infty) = R_0. \tag{3.4}$$

Let s be a fixed real number satisfying $0 \leqq s \leqq m$. We assume that $q(x)$ is a function defined on E^n such that $q = q_1 q_2$ with

$$q_1 \in M_{\alpha,p}, \qquad \alpha < ps - n,$$
$$q_2 \in M_{\beta,p'}, \qquad \beta < p'(m-s) - n, \qquad p' = p/(p-1),$$
$$\int_{|x-y|<1} |q(y)|\, dy \to 0 \qquad \text{as } |x| \to \infty.$$

The operator $A(x, D) + q$ acting on C_0^∞ is preclosed in L^p. Denote it by B_0.

THEOREM 3.2. *The operator B_0 has an intermediate extension B relative to the spaces $H^{s,p}$ and $H^{m-s,p'}$.*

The extension B will be called the s-extension of B_0.

THEOREM 3.3. *Under the above hypotheses,*

$$\Phi_B \supseteq \Phi_A \tag{3.5}$$

and if $\lambda \in \Phi_A$, then

$$i(B - \lambda) = i(A - \lambda). \tag{3.6}$$

THEOREM 3.4. *If, in addition, CR_0 is not empty, then*

$$\Phi_B = \Phi_A \tag{3.7}$$

and (3.6) *holds.*

Another definition of essential spectrum given in [12] adds to $\sigma_{ew}(E)$ all points λ of Φ_E for which $i(E - \lambda) \neq 0$. We denote the essential spectrum according to this definition by $\sigma_{em}(E)$. Thus by Theorems 3.3 and 3.4 we have

$$\sigma_{ew}(B) \subseteq \sigma_{ew}(A), \qquad \sigma_{em}(B) \subseteq \sigma_{em}(A) \tag{3.8}$$

in general, and

$$\sigma_{ew}(B) = \sigma_{ew}(A), \qquad \sigma_{em}(B) = \sigma_{em}(A) \tag{3.9}$$

when $\sigma_{ew}(A)$ is not the whole complex plane.

We now consider the perturbation of $A(x, D)$ by an operator

$$C(x, D) = \sum_{|\mu|<m} C_\mu(x) D^\mu.$$

For $|\mu| \leqq s$ we assume that $C_\mu(x) = C_{1\mu}(x) C_{2\mu}(x)$, where

$$C_{1\mu} \in M_{\alpha-p|\mu|,p}, \qquad C_{2\mu} \in M_{\beta,p'},$$

where $\alpha < ps - n$ and $\beta < p'(m-s) - n$. For $|\mu| > s$ we assume

$$G_{s-|\mu|} \overline{C}_\mu \in M_{\beta,p'},$$

where the operator G_t is given by

$$G_t v = \mathfrak{F}^{-1}(1 + |\xi|^2)^{t/2} \mathfrak{F} v.$$

We also assume

$$\int_{|x-y|<1} |C_\mu(y)| \, dy \to 0 \qquad \text{as } |x| \to \infty, \ |\mu| \leqq s,$$

$$\int_{|x-y|<1} |G_{s-|\mu|} C_\mu(y)| \, dy \to 0 \qquad \text{as } |x| \to \infty, \ |\mu| > s.$$

THEOREM 3.5. *Let L_0 denote the operator $A(x, D) + C(x, D)$ acting on C_0^∞. Under the above hypotheses, it has an s-extension L. Moreover we have*

$$\Phi_L \supseteq \Phi_A \tag{3.10}$$

and if $\lambda \in \Phi_A$, $i(L - \lambda) = i(A - \lambda)$. If CR_0 is not empty, then $\Phi_L = \Phi_A$.

BIBLIOGRAPHY

1 Martin Schechter, *Basic theory of Fredholm operators*, Ann. Scuola Norm. Sup. Pisa **21** (1967), 361–380.

2. F. E. Browder, *Functional analysis and partial differential equations*. II, Math. Ann. **145** (1962), 81–226.

3. Erik Balslev, *The essential spectrum of elliptic differential operators in $L^p(R_n)$*, Trans. Amer. Math Soc. **116** (1965), 193–217.

4. Frantisek Wolf, *On the essential spectrum of partial differential boundary problems*, Comm. Pure Appl. Math. **12** (1959), 211–228.

5. S. G. Mikhlin, *Multidimensional singular integrals and integral equations*, Pergamon Press, Oxford, 1965.

6. Shmuel Agmon, Avron Douglis and Louis Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations*. I, Comm. Pure Appl. Math. **12** (1959), 623–727.

7. A. P. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Math. **24** (1964), 113–190.

8. J. L. Lions, *Une construction d'espaces d'interpolation*, C. R. Acad. Sci. Paris **251** (1960), 1851–1855.

9. Martin Schechter, *Complex interpolation*, Compositio Math.

10. Nachman Aronszajn, Fuad Mulla and Pawel Szeptycki, *On spaces of potentials connected with L^p classes*, Technical Report No. 28, University of Kansas, 1963.

11. Martin Schechter, *On the invariance of the essential spectrum of an arbitrary operator*. II, Ricerche Mat. **16** (1967), 3–26.

12. ———, *On the essential spectrum of an arbitrary operator*. I, J. Math. Anal. Appl. **13** (1966), 205–215.

13. Tosio Kato, *Fractional powers of dissipative operators*, J. Math. Soc. Japan **13** (1961), 246–274.

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ON SELF DUAL L C A GROUPS

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DEFINITION 1.1. Let G be a locally compact Hausdorff Abelian group with a character group G^*. (Hereafter we shall call such groups G as L C A groups.) G is called self dual if there is a topological isomorphism $T: G^* \to G$ from G onto G^*. Some examples of self dual groups are known in literature, but the structure of all such groups is an open problem (see page 423 of [1]). In this note we announce the structure of those self dual groups which are torsion free as abstract Abelian groups. We state some definitions before announcing the main theorem. The complete details will appear elsewhere.

DEFINITION 1.2. Let J be an index set. Let G_α be an L C A group for each $\alpha \in J$. Let $H_\alpha \subset G_\alpha$ be a compact, open subgroup of G_α for every $\alpha \in J$. By the local direct sum G of the groups G_α modulo H_α, we mean the subgroup of $\prod_{\alpha\in J} G_\alpha$ consisting of those elements for which all but a finite number of coordinates lie in H_α. Notice that $H = \prod_{\alpha \in J} H_\alpha$ is contained in G. We topologise G in such a way that H is declared to be open in G, and the relative topology on H as a subspace of G coincides with the product topology of the spaces H_α where H_α is given the relative topology from G_α. We write $G = \sum_{\alpha \in J} G_\alpha$. With this definition G is also an L C A group.

DEFINITION 1.3. Let p be a prime integer >0. Then J_p denotes the field of p-adic numbers with usual addition and topology. With this addition and topology, J_p is a locally compact Abelian group. Any compact, open subgroup H_p of J_p is called p-adic integers. A local direct sum $\sum_{\alpha \in X} G_\alpha$ of the L C A groups G_α is called a canonical p-group if each G_α, where $\alpha \in X$, is isomorphic to p-adic numbers and, in each G_α, some compact open subgroup is fixed in advance.

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