$$\sum_{j=0}^{dr-q} (-1)^j \binom{dr}{j} \left\{ \begin{matrix} \binom{dr-j}{d} \\ r \end{matrix} \right\} = (dr)!/[r!(d!)^r],$$

$$\sum_{j=0}^{dr-q-1} (-1)^j \binom{dr-1}{j} \left\{ \begin{matrix} \binom{dr-j-1}{d} \\ r \end{matrix} \right\} = d(dr)!(r-1)/2[r!(d!)^r], \text{ etc.}$$

Details of proofs, computations, and applications will appear elsewhere.

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FIXED POINTS OF NONEXPANDING MAPS

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Introduction. This paper is concerned with nonexpanding maps from the unit ball of a real Hilbert space into itself. Browder [1] has established that such maps always possess at least one fixed point. We shall develop a method, which resembles the simple iterative method, for approximating fixed points of such maps. In fact, we shall generate a sequence, $\{x_n\}$, by the recursive formula $x_{n+1} = k_{n+1}f(x_n)$ where f is the map in question and $\{k_n\}$ is a sequence of real numbers. Our main result is Theorem 3 which states sufficient conditions on k_n to insure the strong convergence of x_n to a fixed point of f.

Definitions and preliminary observations. Let H be a Hilbert space with inner product denoted by $(\ ,\ )$ and norm by $\|\ \|$. Let B be the unit ball, $B = \{x \in H \mid \|x\| \leqq 1\}$. A map $f\colon B \to B$ is nonexpanding if $\|f(x) - f(y)\| \leqq \|x - y\|$ for all $x, y \in B$.

Assume that $f\colon B \to B$ is nonexpanding. It is not difficult to establish that the set F of fixed points must be convex. Using the convexity of F it is then easy to see that there is at most one $y \in F$ such that $\|y\| = \inf_{z \in F}\|z\|$. Now if $g: B \to B$ is defined by $g(x) = kf(x)$ with $|k| < 1$ then g satisfies $\|g(x) - g(y)\| \leqq |k| \|x - y\|$ for all $x, y \in B$. Consequently there exists a unique point $y_k \in B$ such that $y_k = g(y_k) = kf(y_k)$.

We begin by giving a new proof of a result of Browder [2].

THEOREM 1. *Let $f: B \to B$ be a nonexpanding map and y_k the unique element of B satisfying $y_k = kf(y_k)$ for $|k| < 1$. Then*

$$\lim_{k \to 1;\, |k| < 1} y_k = y$$

where y is the unique fixed point of f with the smallest norm.

PROOF. It is obviously sufficient to show that $y_{k_i} \to y$ as $i \to \infty$ for each sequence k_i, $i = 1, 2, \cdots$, satisfying $k_1 < k_2 < k_3 < \cdots$ and $k_i \to 1$ as $i \to \infty$. The existence of y will be established in the course of the proof.

Assume that $0 < k < l \leqq 1$, $y_k = kf(y_k)$, $y_l = lf(y_l)$ and $d = y_l - y_k$. Then using $\|f(y_l) - f(y_k)\| \leqq \|y_l - y_k\|$ we obtain

$$(l^{-1}(y_k + d) - k^{-1}y_k,\ l^{-1}(y_k + d) - k^{-1}y_k) \leqq \|d\|^2.$$

Thus

$$(l^{-1} - k^{-1})^2\|y_k\|^2 + (l^{-2} - 1)\|d\|^2 \leqq 2(k^{-1} - l^{-1})l^{-1}(y_k, d). \tag{1}$$

Consequently, $(y_k, d) \geqq 0$.

Now note

$$\|y_l\|^2 = (y_k + d, y_k + d) = \|y_k\|^2 + \|d\|^2 + 2(y_k, d)$$

and so

$$\|y_l\|^2 \geqq \|y_k\|^2 + \|y_l - y_k\|^2. \tag{2}$$

The sequence $\|y_l\|$, being monotonic and bounded, converges. Hence $\|y_l - y_k\| \leqq \|y_l\|^2 - \|y_k\|^2 \to 0$ as $k, l \to +\infty$. Thus y_{k_i} converges to some q as $i \to \infty$. Since B is closed $q \in B$. Note that because f is nonexpanding it is continuous. Thus if we take the limit of both sides of (3) below as $i \to \infty$ we find that q is a fixed point of f.

$$y_{k_i} = k_i f(y_{k_i}). \tag{3}$$

Now let p be any fixed point of f. Then $p = 1(f(p))$ and so (2) holds with $p = y_l$, $l = 1$, $y_{k_i} = y_k$ and $k = k_i$ for any $i = 1, 2, \cdots$. Since $y_k \to q$ as $i \to \infty$, $\|y_{k_i}\| \to \|q\|$ as $i \to \infty$ and thus $\|q\| \leqq \|p\|$. Therefore $\|q\| = \inf\|p\|$, p a fixed point of f. As we have noted above there can be no more than one such fixed point of f, which we call y. We have thus shown that $y_{k_i} \to y$ as $i \to \infty$ for each sequence k_i, $i=1, 2, \cdots$ satisfying $k_1 < k_2 < k_3 < \cdots$ and $k_i \to 1$ as $i \to \infty$. This proves the theorem.

Principal results. We know that if we pick any $x_0 \in B$ and define x_n, $n=1, 2, \cdots$, inductively by the formula $x_n = kf(x_{n-1})$ with $|k| < 1$ then $x_n \to y_k$ as $n \to \infty$. We also know that $y_k \to y$ as $k \to 1$, $|k| < 1$. This suggests the following question. What sequences of real numbers $\{k_i\}$, $i=1, 2, \cdots$, have the property that if we define the sequence z_n inductively by

$$z_0 = a, \qquad z_{n+1} = k_{n+1}f(z_n), \qquad n = 0, 1, \cdots, \tag{4}$$

and let y be the fixed point of f with the smallest norm, then

$$z_n \to y \text{ as } n \to \infty \tag{5}$$

irrespective of our choice of Hilbert space H, nonexpanding map $f: B \to B$ where $B = \{x \mid x \in H, \|x\| \leqq 1\}$, and starting point $a \in B$. Such a sequence $\{k_i\}$ will be called *acceptable.*

THEOREM 2. *Three necessary conditions for* $\{k_i\}$, $i=1, 2, \cdots$, *to be acceptable are*

(i) $|k_i| \leqq 1$, *all* $i=1, 2, \cdots$,
(ii) $k_i \to 1$ *as* $i \to \infty$, *and*
(iii) $\prod_{i=1}^{\infty} k_i = 0$.

PROOF. To establish the necessity of a condition on $\{k_i\}$ we need only consider a particular H, f, and a. Take H to be the reals and $a=1$. Then if we set $f(x)=1$ for all $x \in B = \{x \mid |x| \leqq 1\}$ we see that $z_n = k_n$ for $n=1, 2, \cdots$. Condition (i) is necessary in order that $z_n \in B$. This is required so that z_{n+1} is defined which, in turn, is needed for (5) to make sense. In this example $y=1$ and so $z_n \to y$ implies $k_n \to 1$. This shows that (ii) is necessary.

Now take H and a as before and set $f(x) = -x$ for $x \in B$. Then $y=0$ and $z_n = (-1)^n \prod_{i=1}^{n} k_i$. Therefore $z_n \to y$ implies $\prod_{i=1}^{\infty} k_i = 0$. This proves that (iii) is necessary and completes the proof.

THEOREM 3. *Sufficient conditions for* $\{k_i\}$ *to be acceptable are*

(i) $k_i < 1$ *for* $i=1, 2, \cdots$,
(ii) $k_i \leqq k_{i+1}$ *for* $i=1, 2, \cdots$,
(iii) $k_i \to 1$ *as* $i \to \infty$,
(iv) *There exists a sequence* $n(i)$ *such that*
  (a) $n(i+1) \geqq n(i)$ *for* $i=1, 2, \cdots$,
  (b) $\epsilon_{i+n(i)}\epsilon_i^{-1} \to 1$ *as* $i \to \infty$,
  (c) $n(i)\epsilon_i \to \infty$ *as* $i \to \infty$, *where* $\epsilon_i = 1 - k_i$.

PROOF. We let $y_i = k_i f(y_i)$ as above and observe that if $l<n$

$$\begin{aligned}\|z_{l+1} - y_i\| &= \|k_{l+1} f(z_l) - k_i f(y_i)\| \\ &\leqq k_i\|z_l - y_i\| + |k_{l+1} - k_i| \\ &\leqq k_i\|z_l - y_i\| + k_n - k_i.\end{aligned}$$

For $m<n$, set $w_0 = \|z_m - y_i\|$ and $w_{j+1} = k_i w_j + k_n - k_i$. Then an easy induction shows that

$$\|z_{m+j} - y_i\| \leqq w_j = k_i^j\|z_m - y_i\| + (k_n - k_i)(1 + k_i + k_i^2 + \cdots + k_i^{j-1}).$$

Noting that

$$\sum_{l=0}^{j-1} k_i^l \leqq \sum_{l=0}^{\infty} k_i^l = (1 - k_i)^{-1}$$

we obtain the estimate

$$(6) \quad \|z_n - y_i\| \leqq k_i^{n-m}\|z_m - y_i\| + (k_n - k_i)(1 - k_i)^{-1} \quad (m < n).$$

Next we will show that $\|z_{i+n(i)} - y_i\| \to 0$ as $i \to \infty$. Using (6) with $m = i$ and $n = i + n(i)$ we obtain

$$\|z_{i+n(i)} - y_i\| \leqq 2k_i^{n(i)} + (k_{i+n(i)} - k_i)(1 - k_i)^{-1},$$

i.e.,

$$\|z_{i+n(i)} - y_i\| \leqq 2k_i^{n(i)} + (\epsilon_i - \epsilon_{i+n(i)})\epsilon_i^{-1}.$$

By condition (iv)(b) we have

$$(\epsilon_i - \epsilon_{i+n(i)})\epsilon_i^{-1} = 1 - \epsilon_{i+n(i)}\epsilon_i^{-1} \to 0 \quad \text{as } i \to \infty.$$

Since $\epsilon_i \to 0$ as $i \to +\infty$,

$$\log(k_i^{n_i}) = n(i)\log(1 - \epsilon_i) \to -\infty \quad \text{as } i \to \infty$$

if $n(i)\epsilon_i \to \infty$ as $i \to \infty$. But this is condition (iv)(c). Therefore

$$(7) \qquad \|z_{i+n(i)} - y_i\| \to 0 \quad \text{as } i \to \infty.$$

-Since $n(i+1) \geqq n(i)$, all i, for any n sufficiently large there exists a unique j such that $j + n(j) \leqq n < j + 1 + n(j+1)$. This j increases without bound as $n \to \infty$. Now using (6) with $m = j + n(j)$ and $i = j + 1$ we obtain

$$\begin{aligned}\|z_n - y_{j+1}\| &\leqq k_{j+1}^{n-j-n(j)} \|z_{j+n(j)} - y_{j+1}\| + (k_n - k_{j+1})(1 - k_{j+1})^{-1} \\ &\leqq 1(\|z_{j+n(j)} - y_j\| + \|y_j - y\| + \|y - y_{j+1}\|) \\ &\quad + (\epsilon_{j+1} - \epsilon_{j+1+n(j+1)})\epsilon_{j+1}^{-1}.\end{aligned}$$

It now follows from (7), Theorem 1, and condition (iv)(b) that $\|z_n - y_{j+1}\| \to 0$ as $n \to \infty$. But $y_{j+1} \to y$ as n and thus j approaches infinity. Therefore $z_n \to y$ as $n \to \infty$. Q.E.D.

COROLLARY. *If* $k_i = 1 - i^{-x}$ *with* $0 < x < 1$, *then* $\{k_i\}$ *is acceptable.*

PROOF. Pick r such that $x < r < 1$. Then set $n(i) = [i^r]$ where $[t]$ is the greatest integer equal to or less than t. It is easy to verify that $\{k_i\}$ and $n(i)$ satisfy conditions (i)–(iv) of Theorem 3 once one notes that $n(i)i^{-r} \to 1$ as $i \to \infty$.

THEOREM 4. *Let* $\{k_i\}$ *and* $\{m_i\}$ *satisfy* $|k_i| < 1$, $k_i \to 1$, *and* $k_i^{m_i} \to 0$. *If* $x_0 \in B$ *and* x_n *are defined inductively by*

$$x_n = k_n f(k_n f \cdots (k_n f(x_{n-1})) \cdots) \quad n = 1, 2, \cdots \quad m_n \text{ times},$$

then $x_n \to y$ *as* $n \to \infty$.

PROOF. An easy induction on the following inequality (8) for $q \in B$

$$\| k_n f(q) - y_{k_n}\| = \|k_n f(q) - k_n f(y_{k_n})\| \leqq k_n \|q - y_{k_n}\| \tag{8}$$

yields

$$\|x_n - y_{k_n}\| \leqq k_n^{m_n} \|x_{n-1} - y_{k_n}\| \leqq 2k_n^{m_n}. \tag{9}$$

Thus $\|x_n - y_{k_n}\| \to 0$ as $n \to \infty$ and since $y_{k_n} \to y$ as $n \to \infty$, $x_n \to y$ as $n \to \infty$. Q.E.D.

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