NONLINEAR EQUATIONS OF EVOLUTION AND NONLINEAR ACCRETIVE OPERATORS IN BANACH SPACES

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Introduction. Let X be a real Banach space, T a (possibly) nonlinear mapping with domain $D(T)$ in X and range $R(T)$ in X. Following [7] and [8], we shall say that T is *accretive* if for all x and y in $D(T)$,

$$(T(x) - T(y), J(x - y)) \geqq 0, \tag{1}$$

where (assuming that the conjugate space X^* of X is strictly convex), J is the mapping of X into X^* which assigns to each x of X the bounded linear functional $w = J(x)$ such that $(w, x) = \|w\| \cdot \|x\|$ and $\|w\| = \|x\|$.

It is our object in the present note to present some new and sharper results on two related topics:

(1) The existence theory of solutions for the initial value problem for nonlinear equations of evolution of the form

$$du/dt + T(t)u(t) = f(t, u(t)) \quad (t \geqq 0) \tag{2}$$

with the initial condition $u(0) = x_0$.

(2) The existence theory of solutions of the equation

$$T(u) = w, \tag{3}$$

for an accretive operator T and an element w of X.

In the study of the equation of evolution (2), we assume that for each t in R^+, $T(t)$ is an accretive operator such that $D(T(t))$ is independent of t and $R(T(t)+I) = X$, while f is a continuous, bounded mapping of $R^+ \times X$ into X. For the special case that X is a Hilbert space and $T(t)$ is linear, such results were obtained in Browder [1] and Kato [12], and extensions for $T(t)$ linear and more general Banach spaces X were given in Murakami [17] and Browder [7] (and in an earlier version of [8]). Results for $T(t)$ nonlinear were first obtained by Komura [15] in Hilbert space and extended to more general Banach spaces by Kato [13] for the case in which $f = 0$. Our proofs (which are given in detail in [8]) and those of Kato [13] are based upon an elementary device applied by Murakami in [17].

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The connection between the equation of evolution (2) and the nonlinear functional equation (3) is based upon a new way of relating the theory of nonlinear accretive operators with the fixed point theory of nonexpansive nonlinear mappings in uniformly convex Banach spaces (Browder [7], Göhde [10], Kirk [14]). In the more restricted context of Hilbert spaces (or more generally of Banach spaces having weakly continuous duality mappings), the writer in [3], [4], and [6] derived a general fixed point theory for nonexpansive mappings from the existing theory of monotone operators. In the present situation, we reverse this connection and derive a general mapping theory for nonlinear accretive operators from the fixed point theory of nonexpansive mappings in uniformly convex spaces. The force of this general theory is substantially equal to that of the earlier theory of monotone operators in Hilbert space and gives stronger results than those outlined in [7].

In a following note, we shall show on the basis of our new general theory for accretive operators that the relationship between accretive and nonexpansive operators may again be reversed and new and still stronger fixed point theorems for mappings of the nonexpansive type can be derived from the new results on accretive operators.

The writer should like to thank Prof. K. Jörgens for making him acquainted with the paper [15] by Komura and Prof. T. Kato for a preprint of his paper [13].

We remark that for the case of linear operators $T(t)$, the general theory presented below includes, as a special case for Banach spaces X with X^* uniformly convex, the classical Hille-Yosida theory of contraction semigroups ([11], [19]).

1. Nonlinear equations of evolution. In the present section, we present without proof the basic results on nonlinear equations of evolution in Banach spaces mentioned above. (The detailed arguments for these theorems are given in [8].)

THEOREM 1. *Let X be a real Banach space whose conjugate space X^* is uniformly convex, $R^+ = \{t \mid t \in R^1,\ t \geqq 0\}$. For each t in R^+, let $T(t)$ be a (possibly) nonlinear mapping with domain $D(T(t))$ in X which is independent of t such that $T(t)$ is accretive and the range of $T(t)+I$ is all of X for each t in R^+. Let f be a mapping of $R^+ \times X$ into X which is demicontinuous (i.e. continuous from $R^+ \times X$ in the strong topology to the weak topology of X) and such that f maps bounded sets in $R^+ \times X$ into bounded sets in X. Suppose further that $T(t)$ and $f(t, u)$ satisfy all of the following three conditions:*

(a) *There exists a continuous function c_1 from $R^+ \times R^+ \times R^+$ to R^+*

such that for each pair t and s in R^+ *and each element u in* $D(T(t))$, *we have*

$$(4)\quad \|T(t)u - T(s)u\| \leqq |t - s| \cdot c_1(t, s, \|u\|)(1 + \|T(t)u\| + \|T(s)u\|).$$

(b) *There exists a continuous function* c_2 *from* R^+ *to* R^1 *such that for all t in* R^+ *and all u and v in X*,

$$(5)\quad (f(t, u) - f(t, v), J(u - v)) \leqq c_2(t)\|u - v\|^2.$$

(c) *There exists a continuous function* c_3 *from* $R^+ \times R^+ \times R^+$ *to* R^+ *such that for each pair t and s in* R^+ *and each u in* $D(T(t))$,

$$(6)\quad \|f(t, u) - f(t, v)\| \leqq m|t - s| c_3(t, s, \|u\|)(1 + \|T(t)u\| + \|T(s)u\|).$$

Then,

(I) *For each* u_0 *in* $D(T(0))$, *there exists one and only one strongly continuous, once weakly continuously differentiable function u from* R^+ *to X such that* $u(t)$ *lies in* $D(T(t))$ *for each t in* R^+ *and satisfies the nonlinear equation of evolution*

$$(2)\quad (du/dt)(t) + T(t)u(t) = f(t, u(t)) \quad (t \in R^+)$$

with the initial condition

$$(7)\quad u(0) = u_0.$$

(II) *If we denote the corresponding solution of the equation* (2) *on the interval* $[s, \infty)$ *with the initial condition* $u(s) = u_0$ *by* $u(t) = U(t, s)u_0$, *then for each t and s in* R^+ *with* $s \leqq t$, $U(t, s)$ *is a Lipschitzian mapping of X into X with Lipschitz norm bounded by*

$$(8)\quad \|U(t, s)\|_{\mathrm{Lip}} \leqq \exp\left(\int_s^t 2c_2(r)\,dr\right).$$

(III) *There exists a continuous function* c_4 *from* $R^+ \times R^+$ *to* R^+ *which depends only on the functions* c_1, c_2, *and* c_3 *such that for any solution* $u(t)$ *on* R^+ *of the equation* (2) *with initial condition* u_0 *in* $D(T(0))$, *we have*

$$(9)\quad \|(du/dt)(t)\| \leqq c_4(t, \|(du/dt)(0)\|).$$

THEOREM 2. *Let* $T(t)$ *and* $f(t, u)$ *satisfy all the hypotheses of Theorem* 1 *except the Lipschitz condition* (c) *on* $f(t, u)$ *in t. Then for each* u_0 *in* $D(T(t))$, *there exists a sequence of strongly continuous functions* $\{u_n\}$ *from* R^+ *to X such that each* u_n *is once weakly continuously differentiable from* R^+ *to X*, $u_n(t)$ *lies in* $D(T(t))$ *for each t in* R^+, $u_n(0) = u_0$ *for every n, and* u_n *converges to a mild solution u of the equation* (2) *on* R^+, *i.e.,* $u_n(t)$ *converges strongly to* $u(t)$ *uniformly on bounded intervals of* R^+ *and*

$$du_n/dt + T(t)(u_n(t)) - f(t, u_n(t)) \to 0$$

weakly in X, *uniformly on each bounded interval of* R^+.

Let us connect these results with the theory of nonlinear semigroups of nonexpansive mappings $U(t)$ of X into X, i.e.,

$$U(t+s) = U(t)U(s) \quad (t, s \in R^+),$$
$$U(0) = I, \; U(t)u - u \to 0 \text{ strongly in } X \text{ as } t \to 0^+ \; (u \in X).$$

The infinitesimal generator T of such a semigroup $U(t)$ is defined by

$$T(u) = \lim_{t \to 0^+} \{(U(t)u - u)/t\}$$

whenever this limit exists in the weak topology of X.

THEOREM 3. *Let X be a Banach space with X^* uniformly convex. Then,*

(a) *If T is an accretive operator with dense domain such that $T+I$ maps onto X, then $(-T)$ is the infinitesimal generator of a one-parameter semigroup of nonexpansive nonlinear operators in X.*

(b) *Let T be a mapping with domain and range in the general Banach space X such that $(-T)$ is the infinitesimal generator of a one-parameter semigroup of nonexpansive mappings of X. Suppose further that for each w_0 in X, $T_{w_0}(u) = -T(u) + w_0$ is the infinitesimal generator of a semigroup of nonlinear mappings in X. Then T is accretive and $T+I$ has all of X as its range.*

THEOREM 4. *Let X be a Banach space with strictly convex dual space X^*. A necessary and sufficient condition that a mapping T with domain $D(T)$ in X and values in X satisfy the two conditions that T be accretive and that $T+I$ map onto X, is the following:*

(i) *For every $r > 0$, $T + rI$ has an inverse defined on all of X which is Lipschitzian and satisfies the bound*

$$\| (T + rI)^{-1} \|_{\mathrm{Lip}} \leq 1/r. \tag{10}$$

For linear mappings T, T is a weak infinitesimal generator if and only if T is a strong infinitesimal generator, and the Hille-Yosida theorem can be derived as a corollary of Theorems 3 and 4.

2. **Nonlinear accretive operators.** Combining the existence theorem, Theorems 1 and 3, for nonlinear semigroups with the fixed point theory of nonexpansive mappings in uniformly convex spaces, we obtain the following general existence theorem for solutions of nonlinear functional equations involving accretive operators in Banach spaces.

THEOREM 5. *Let X be an uniformly convex Banach space with its conjugate space X^* also uniformly convex, and let T and T_0 be two accretive mappings with domain and range in X. Suppose that*

(a) *The range of $T+I$ is all of X. $D(T)$ is dense in X.*

(b) *T_0 is defined and demicontinuous on all of X and maps bounded subsets of X into bounded subsets of X.*

(c) *The mapping $T+T_0$ defined with domain $D(T)$ satisfies the condition that*

$$\|(T + T_0)(u)\| \to +\infty, \quad \text{as } \|u\| \to +\infty \quad (u \in D(T)).$$

Then, the range of $(T+T_0)$ is all of X, i.e., for each w in X, there exists an element u in $D(T)$ such that

$$T(u) + T_0(u) = w.$$

Before proceeding to the proof of Theorem 5, we make two remarks upon the hypotheses:

REMARK 1. A stronger hypothesis than the asymptotic condition (c) of Theorem 5, which therefore implies condition (c), is that $(T+T_0)$ is coercive in the sense of [7], i.e., that

$$((T + T_0)(u), Ju)/\|u\| \to +\infty \quad (u \in D(T))$$

as $\|u\| \to +\infty$.

REMARK 2. The hypothesis that X^* is uniformly convex is equivalent by a classical result of Smulian to the condition that X itself have uniformly C^1-norm, i.e., that the duality mapping J is defined as a single valued mapping of X into X^* and is uniformly continuous in the strong topologies on the unit sphere of X.

PROOF OF THEOREM 5. Since we may subtract a fixed element y_0 of X from the mapping T_0 without affecting the validity of the hypotheses of Theorem 5 and since thereby we should replace $(T+T_0)(u)$ by $(T+T_0)(u)-y_0$, it follows that to show that $R(T+T_0)=X$, it suffices to show that 0 lies in $R(T+T_0)$.

Under the hypotheses of Theorem 5, we may apply the results of Theorems 1 and 3 to obtain the conclusion that the operator $-(T+T_0)$ is the infinitesimal generator of a semigroup of (nonlinear) nonexpansive operators in the Banach space X in the following precise sense: For each t in R^+, there exists a (nonlinear) mapping $U(t)$ of X into X such that

(1) For all u and v in X, $t \in R^+$,

$$\|U(t)u - U(t)v\| \leqq \|u - v\|.$$

(2) For all s and t in R^+, u in X, we have

$$U(t+s) = U(t)U(s), \quad U(0) = I, \quad U(t)u \to u \quad \text{as } t \to 0^+.$$

(3) For u in $D(T) = D(T+T_0)$,

$$U(t)u \in D(T + T_0), \quad t \geqq 0,$$

(4) For u in $D(T+T_0)$,

$$\text{weak} \lim_{h\to 0} h^{-1}(U(h) - I)u = -(T + T_0)(u).$$

(5) If the weak limit as $h \to 0^+$ of $h^{-1}(U(h)u - u)$ exists, then u lies in $D(T+T_0) = D(T)$.

We shall derive our desired conclusion upon the mapping $(T+T_0)$ by using properties of the nonlinear semigroup $\{U(t)\}$.

We fix v_0 in $D(T)$, and consider the orbit $v(t) = U(t)v_0$ under the semigroup. For any $h > 0$, $v(t+h) = U(t)(U(h)v_0)$ is the orbit of the semigroup beginning at $U(h)v_0$. By the nonexpansive property of the mapping $U(t)$, it follows that

$$\|v(t + h) - v(t)\| = \|U(t)U(h)v_0 - U(t)v_0\| \leqq \|U(h)v_0 - v_0\|.$$

By property (3) of the semigroup, $v(t)$ lies in $D(T+T_0)$ for all $t \geqq 0$. Hence by property (4) above,

$$-(T + T_0)(v(t)) = \text{weak} \lim_{h\to 0^+} h^{-1}(v(t + h) - v(t)),$$

so that

$$\begin{aligned}\|(T - T_0)(v(t))\| &\leqq \liminf_{h\to 0^+} \|h^{-1}(v(t + h) - v(t))\| \\ &\leqq \liminf_{h\to 0^+} \|h^{-1}(U(h)v_0 - v_0)\|.\end{aligned}$$

As $h \to 0^+$,

$+h^{-1}(U(h)v_0 - v_0) \rightharpoonup -(T + T_0)(v_0)$ ($\rightharpoonup$ denotes weak convergence).

Therefore there exists a constant $M > 0$ such that

$$\liminf_{h\to 0^+} \|h^{-1}(U(h)v_0 - v_0)\| \leqq M,$$

where M depends upon v_0 and the semigroup $\{U(t)\}$. Hence for all t in R^+,

$$\|(T + T_0)(v(t))\| \leqq M.$$

By the hypothesis (c) of Theorem 5 which asserts that $\|(T+T_0)(u)\| \to +\infty$ as $\|u\| \to +\infty$, it follows that there exists another constant $M_1 > 0$ such that

$$\|v(t)\| \leqq M_1, \quad t \geqq 0.$$

Let $R = 2M_1$, and consider the set

$$K_s = \bigcap_{t \geqq s} B_R(v(t)),$$

where for any u in X, $B_R(u)$ is the closed ball of radius R about u. Each K_s is a closed convex subset of X, contains v_0, and is itself contained in the ball of radius $3M_1$ about v_0. Moreover K_s is nondecreasing with increasing s in R^+. Hence

$$K = \lim_{s \to +\infty} K_s = \bigcup_{s \in R^+} K_s$$

is a bounded convex subset of X.

If u lies in K_s and t lies in R^+, then for $r \geqq s+t$,

$$\|U(t)u - v(r)\| = \|U(t)u - U(t)v(r-t)\| \leqq \|u - v(r-t)\| \leqq R,$$

i.e., $U(t)u$ lies in K_{s+t}. It follows that if u lies in K, $U(t)$ lies in K for all $t \geqq 0$, i.e., K is invariant under all the mappings of the semigroup $\{U(t)\}$. Letting K_1 be the closure of K in X, it follows that K_1 is a closed bounded convex subset of the uniformly convex Banach space X which is mapped into itself by each mapping $U(t)$ of the commuting family of nonexpansive mappings $\{U(t)\}$. We may apply Theorem 2 of [5] and conclude that there exists a point u_0 of K_1 which is a common fixed point of all the mappings $U(t)$, $t \geqq 0$, i.e., for which

$$U(t)u_0 = u_0, \quad t \geqq 0.$$

For this point u_0, however, we see that for any $h > 0$,

$$h^{-1}(U(h)u_0 - u_0) = 0.$$

Hence, the limit of these difference quotients exists and equals 0. By property (5) of the semigroup, u_0 lies in $D(T+T_0)$. By property (4),

$$(T + T_0)(u_0) = - \lim_{h \to 0^+} h^{-1}(U(h) - I)(u_0) = 0. \qquad \text{q.e.d.}$$

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