RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited. Manuscripts more than eight typewritten double spaced pages long will not be considered as acceptable.

A CONNECTION BETWEEN α-CAPACITY AND $m-p$ POLARITY

BY WALTER LITTMAN[1]

Communicated by J. Moser, July 2, 1967

In investigating questions of removable singularities of second order partial differential equations, Serrin [3] introduced the notion of the M_s capacity of a compact set S in R^n as follows:

$$M_s(S) = \inf \int |\operatorname{grad} u|^s dx$$

where the infimum is taken over all continuously differentiable functions u having compact support in R^n and $\geqq 1$ on S. It turns out that in this definition the "$\geqq$" sign may be replaced by an equal sign. Indeed it is the definition using the equal sign which is really made use of in the proofs. The equivalence of the two definitions is, roughly speaking, due to the fact that, in taking the infimum, each competing function u may be truncated, i.e., be replaced by $\bar{u} \equiv \min (u, 1)$. $\bar{u}$ will not in general be continuously differentiable, but this difficulty can be overcome.

There is a more classical notion of capacity, $C_\alpha(S)$ due to Frostman and others. (See [4], for example, for a brief description of the relevant properties of C_α.) Wallin [4] has exhibited a close relationship between $M_s(S)=0$ and $C_\alpha(S)=0$ for appropriately related values of α and s.

In order to investigate certain questions of removable singularities for higher order partial differential equations, the author [1] has found that the appropriate concept for the smallness of a set was that of $m-p$ polarity. We define, for a compact set S in R^n,

$$M_{m,p}(S) \equiv \inf |u|_{m,p}^p \tag{1}$$

[1] This research was partially supported by U.S.A.F. Grant AFOSR 883-65.

where the infimum is taken over all C^∞ functions with compact support in R^n which are *identically equal to one* near S. Here $|u|_{m,p}^p$ is the pth power of the Sobolev norm, i.e., the sum of the pth powers of the function u and all its derivatives of u of orders $\leqq m$. S is said to be *$m-p$ polar* if $M_{m,p}(S)=0$.

Unfortunately the truncation argument does not work for $m>1$ and the condition that $u\equiv 1$ (near S) cannot in general be replaced by $u\geqq 1$. Thus we introduce another notation to denote the infimum in (1) taken instead over all C^∞ functions with compact support in R^n which $\geqq 1$ near S. *We denote that infimum by* $N_{m,p}(S)$. It is not difficult to see that $M_s(S)=0$ is equivalent to $N_{1,s}(S)=0$.

Although Wallin's method can be extended to higher order derivatives (as is stated in [4]) it results in relationships between $N_{m,p}(S)=0$ and $C_\alpha(S)=0$, and yields no sufficient condition for S to be $m-p$ polar, in terms of $C_\alpha(S)=0$. The aim of this note is to establish such a condition.

Our main result is the following theorem.

THEOREM 1. *Let S be a compact set in R^n. Suppose $n-mp\geqq 0$. Then $C_{n-mp}(S)=0$ implies*

$$M_{m,p}(S) = 0 \quad \text{if } 2 \leqq p < \infty,$$
$$M_{m,p-\epsilon}(S) = 0 \quad \text{for every } \epsilon > 0 \text{ if } 1+\epsilon \leqq p < 2.$$

A partial converse is given by

THEOREM 2. *Let S be a compact set in R^n. Suppose $n-mp\geqq 0$. Then $M_{m,p}(S)=0$ implies*

$$C_{n-mp}(S) = 0 \quad \text{if } 1 \leqq p \leqq 2,$$
$$C_{n-mp+\epsilon}(S) = 0 \quad \text{for every } \epsilon > 0 \text{ if } 2 < p < \infty.$$

Before proceeding with proofs, let us see how these theorems are related to questions of removable singularities. Suppose V is an open set in R^n and S a compact set contained in V. Let L be a linear partial differential operator defined by

$$Lu = \sum_{|\alpha|\leqq m} D^\alpha(a_\alpha(x)u),$$

where the a_α are bounded measurable functions defined in a domain V. If $u\in L^p(V)$ being a weak solution to the mth order linear partial differential equation $Lu=0$ in $V-S$ implies that u is a weak solution in all of V, we say that *S is removable with respect to* (L, V, L^p). It was proved in [1] that if L is strongly elliptic and satisfies the unique continuation property in a slightly larger domain (the last requirement can be eliminated and the first weakened) then S being removable with respect to (L, V, L^p) is equivalent to S being $m-p'$ polar, where $1/p+1/p'=1$, $1<p<\infty$. (For the exact statements see [1].) Thus we obtain the following theorems:

THEOREM 3. *Suppose $n-mp'\geqq 0$; then*

$$C_{n-mp'}(S) = 0$$

implies that S is removable with respect to

$$(L,\, V,\, L^p) \quad \text{if } 1 < p \leqq 2, \qquad (L,\, V,\, L^{p+\epsilon}) \quad \text{if } 2 \leqq p < \infty.$$

THEOREM 4. *Suppose the a_α are uniformly Hölder continuous in $W\supset \overline{V}$. Suppose L is strongly elliptic and satisfies the unique continuation property in W. Then if S is removable with respect to (L, V, L^p) it follows that*

$$C_{n-mp'}(S) = 0 \quad \text{if } 1 < p' \leqq 2,$$
$$C_{n-mp'+\epsilon}(S) = 0 \quad \text{for every } \epsilon > 0 \quad \text{if } 2 \leqq p' < \infty.$$

NOTE. The strong ellipticity condition can be weakened and the unique continuation property can be eliminated.

Before proceeding with the proof of Theorem 1 we shall state a few lemmas.

LEMMA 1 (NIRENBERG [2]). *Let $u\in C_0^\infty(R^n)$. Then, for $j<m$, $1\leqq r \leqq \infty$, $1\leqq q\leqq \infty$,*

$$\left| D^j u \right|_p \leqq \text{Constant} \left| D^m u \right|_r^{j/m} \left| u \right|_q^{1-j/m}.$$

Here the constant is independent of u; j, m are positive integers; subscripts indicate L^p norms; and $|D^j u|_p$ denotes the maximum of the L^p norms of all jth order derivatives of u. An alternate form of this inequality is

$$\left| D^{m\theta} u \right|_p \leqq \text{Constant} \left| D^m u \right|_r^{\theta} \left| u \right|_q^{1-\theta},$$

where $0<\theta<1$, $m\theta$ is integral, and $1/p=\theta/r+(1-\theta)/q$. In particular,

$$\left| D^{m\theta} u \right|_{r/\theta} \leqq \text{Constant} \left| D^m u \right|_r^{\theta} \left| u \right|_\infty^{1-\theta}.$$

LEMMA 2. *Suppose there exists a sequence of functions $u_\nu\in C_0^\infty(R^n)$ such that the u_ν are uniformly bounded in R^n,*

$$\left| u_\nu \right|_{m,p} \to 0 \quad \text{as } \nu \to \infty, \qquad u_\nu \geqq 1 \text{ near } S \text{ (a compact set).}$$

Then there exists another sequence of functions $w_\nu \in C_0^\infty(R^n)$ such that w_ν is uniformly bounded in R^n,

$$|w_\nu|_{m,p} \to 0 \quad \text{as } \nu \to \infty, \qquad w_\nu \equiv 1 \text{ near } S.$$

PROOF. Let $f(t)$ be a C^∞ function of the single real variable t such that

$$f(t) = 0 \quad \text{for } t \leqq 0, \quad 0 < f(t) < 1 \quad \text{for } 0 < t < 1, \quad f(t) = 1 \quad \text{for } t \geqq 1.$$

We shall eventually take $w_\nu \equiv f(u_\nu)$. We must estimate

$$\int |D^m f(u)|^p dx \quad \text{or} \quad |D^m f(u)|_p.$$

We have

$$\begin{aligned} Df(u) &= f' \cdot Du, \\ D^2 f(u) &= f'' \cdot (Du)^2 + f' D^2 u, \\ D^k f(u) &\simeq \sum_{j=1}^{k} \Big(f^{(j)} \cdot \prod_{\Sigma \nu_i = k} (D^{\nu_i} u) \Big). \end{aligned}$$

In the last formula the $\simeq$ sign needs some interpretation. It means: A partial derivative of $f(u)$ regarded as a function of x of order k is the sum of terms each of which is a product of one of the functions $f^{(j)}(u)$ $(j=1, 2, \cdots, k)$ and a certain number of partial derivatives $D^{\nu_i} u$ of u (with respect to the x's) such that the sum of all the orders of the partial derivatives occurring as factors equals k. (The same partial derivative may, of course, occur several times as a factor.) It is readily seen that the last formula, interpreted in the manner described, follows easily by induction.

Now to estimate $|D^m f(u)|_p^p$. Since the functions $f^{(j)}$ are bounded it suffices to estimate

$$\Big| \prod_{\Sigma \nu_i = m} D^{\nu_i} u \Big|_p^p.$$

By Hölder's inequality, the above does not exceed

$$\prod_{\Sigma \nu_i = m} |(D^{\nu_i} u)^p|_{p_i} = \prod_{\Sigma \nu_i = m} |D^{\nu_i} u|_{pp_i}^p,$$

where $\sum 1/p_i = 1$, $p_i \geqq 1$.

If we choose the p_i such that $p_i \nu_i = m$, then, using Lemma 1, we see that

$$| D^{\nu_i} u |_{pp_i} \leqq \text{Const} | D^m u |_p^{1/p_i} | u |_\infty^{1-1/p_i}.$$

Since $\sum \nu_i = m$ and since the ν_i are positive integers, the number of ν_i cannot exceed m. Hence

$$|\prod (D^{\nu_i} u) |_p \leqq \text{Const } | u |_{m,p} \cdot (1 + | u |_\infty^{m-1}),$$

from which it follows that

$$| f(u) |_{m,p} \leqq \text{Const } | u |_{m,p} [1 + | u |_\infty^{m-1}].$$

Letting $u = u_\nu$ and $f(u_\nu) \equiv w_\nu$ we see that w_ν is the desired sequence. This proves Lemma 2.

PROOF OF THEOREM 1. Suppose the hypotheses of the theorem are satisfied for S. Then, in view of Lemma 2, it suffices to construct a sequence of functions $u_\nu \in C_0^\infty$ such that $u_\nu \geqq 1$ near S, the u_ν are uniformly bounded in R^n and

$$| u_\nu |_{m,p} \to 0 \quad \text{as } \nu \to \infty \quad \text{for } p \geqq 2,$$
$$| u_\nu |_{m,p-\epsilon} \to 0 \quad \text{as } \nu \to \infty \quad \text{for } 1 + \epsilon \leqq p < 2.$$

The existence of such a sequence is proved by Wallin in [4] for $m = 1$. The same method also yields the proof for other values of m. We shall not reproduce the details here.

To prove Theorem 2 we note that it suffices to prove it with $M_{m,p}(S)$ replaced by $N_{m,p}(S)$. For $m = 1$ this is proved in [4], where the proof can be generalized to include other m.

BIBLIOGRAPHY

1. W. Littman, *Polar sets and removable singularities of partial differential equations*, Ark. Mat. **7** no. 1 (1967), 1–9.

2. L. Nirenberg, *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa Ser. III, Vol. **13** (1959) pp. 123–131 (Lecture 2).

3. J. Serrin, *Local behavior of solutions of quasilinear equations*, Acta Math., **111** (1964), 247–302.

4. H. Wallin, *A connection between α capacity and L^p classes of differentiable functions*, Ark. Mat. **5** (1964), 331–341.

UNIVERSITY OF MINNESOTA